LECTURE 5

31.08.2020

KAROLINE MOE

DRE 7017

1. OPTIMIZATION PROBLEMS
2. UNCONSTRAINED OPTIMIZATION

FMEA 3.1 - 3.2

ME 17

(S 2, 4, 7.3 - 7.6, 8.4 - 8.7)

① OPTIMIZATION

Want to find ~~the~~ maximum or minimum of an objective function $f(x_1, .., x_n) : D \rightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^n$ the admissible set / constraint set.

optimize

$$\max(\min)\ f(\underline{x}) \text{ subject to } \underline{x} \in D.$$

UNCONSTRAINED & CONSTRAINED OPTIMIZATION

Classical case: The optimum occurs at an interior point of D.

Lagrange problem: D is the set of all points that satisfy a given system of equations, and we maximize a function subject to equality constraints

Nonlinear programming problem: D consists of all points that satisfy a given system of inequality constraints.

Necessary Kuhn Tucker conditions
Sufficient conditions

Want to find
DEF:

Assume $\underline{x}^* \in D$ s.t. $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x} \in D$:

$\underline{x}^*$ is called (global) maximum point (strict if >)

$f(\underline{x}^*)$ is called maximum value

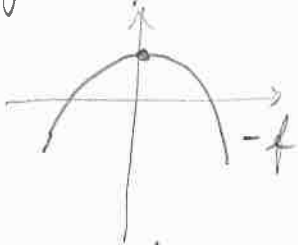
- similar for minimum: $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x} \in D$.
- max/min are called extreme points & extreme values

NOTE:

① $\underset{\underline{x}\in D}{\overset{\text{argument}}{\arg}\max} f(\underline{x}) := \{\underline{x}^*\in D \mid \forall\, \underline{x}\in D,\ f(\underline{x}^*) \geq f(\underline{x})\}$

- Empty
- One element
- several elements

② As usual: $\arg\min f(\underline{x}) = \arg\max -f(\underline{x})$

③ $\underset{\underline{x}\in D}{\arg\max} f(\underline{x})$ has solutions $\iff \sup f(D) \in f(D)$

$f(D) = \{f(\underline{x}) \mid \underline{x}\in D\} \subseteq \mathbb{R}$

The attainable values of $\underset{\underline{x}\in D}{\arg\max} f(\underline{x})$

USEFUL LEMMA:

If $\varphi : \mathbb{R}\to\mathbb{R}$ is a strictly increasing function,

$x < y \Rightarrow f(x) < f(y)$, then

$\underset{\underline{x}\in D}{\arg\max} f(\underline{x})$ has the same solutions as $\underset{\underline{x}\in D}{\arg\max} \varphi(f(\underline{x}))$

$$\arg\max f(\underline{x}) = \arg\max \varphi(f(\underline{x}))$$

EX: $f(x,y) = \sqrt{x^2+y^2}$ $\qquad \varphi(u) = u^2$

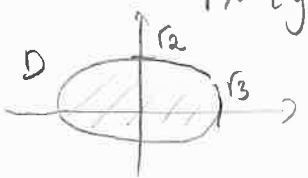
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ $\qquad \varphi : \mathbb{R}^+ \to \mathbb{R}^+$

on $2x^2+3y^2 \leq 6$ $\qquad \varphi'(u) = 2u \geq 0$ strictly increasing for $u\in(0,\infty)$

$D = \{(x,y) \mid 2x^2+3y^2 \leq 6\}$

So $\underset{(\min)}{\arg\max}\ \underset{\sqrt{x^2+y^2}}{\underset{\parallel}{f(x,y)}} = \underset{(\min)}{\arg\max}\ \underset{x^2+y^2}{\underset{\parallel}{g(x,y)}}$

on



Maple

RECALL:

Extreme value thm: f continuous, D compact $\Rightarrow$ $f(D)$ compact and has both max and min in D.

f not continuous or D not compact $\Big\}$ May or may not have solutions to optimization prob.

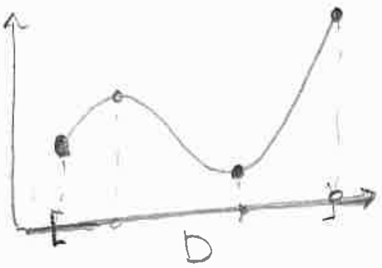
DEF: A maximizer (or minimizer) $\underline{x}^* \in D$ of $f(\underline{x})$ is called
UNCONSTRAINED if $\underline{x}^* \in D^\circ = D \setminus \partial D$ (interior pt)
CONSTRAINED if $\underline{x}^* \in \partial D$ (boundary pt)

CASES:

① $D = \mathbb{R}^n$ or $D \subseteq \mathbb{R}^n$ is open
Only unconstrained extreme points.

② $D \subseteq \mathbb{R}^n$ not open:
Can have both constrained and unconstrained extreme points.

MOREOVER:



We are interested in local extreme points as well as global extreme pts.

DEF: • $\underline{x}^* \in D$ is called a local max for $f(\underline{x})$ if there is an open ball $B(\underline{x}^*, \epsilon)$, such that $f(\underline{x}) \le f(\underline{x}^*)$ for all $\underline{x} \in B(\underline{x}^*, \varepsilon)$.

• Similar for local min

NOTE: $\underline{x}^*$ maximum (minimum) for f $\Rightarrow$ $\underline{x}^*$ local maximum (minimum) for f

DEF: Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, f continuous.

① A STATIONARY POINT for f is a point $\underline{x}^* \in D$ s.t. $Df(\underline{x}^*) = \underline{0}$; i.e.,
f is differentiable and $\underbrace{\frac{\partial f}{\partial x_1}(\underline{x}^*) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}^*) = 0}_{\text{FOC (first order conditions)}}$.

② A CRITICAL POINT for f is a point $x^* \in D$ such that $Df(\underline{x}^*) = 0$ or $Df(\underline{x}^*)$ does not exist.
(If f is C^1, then {critical pts} = {stationary pts}.

RESULT: If $\underline{x}^*$ is a local max for the optimization problem $\arg\max_{\underline{x} \in D} f(\underline{x})$, then one of the following holds:

① $\underline{x}^*$ is a critical interior pt of D.

OR

② $\underline{x}^*$ is a boundary point of D.

SUMMING IT UP:

Want to find $\arg\max_{x \in D} f(x)$ for f continuous.

- Consider FOC $\left.\begin{array}{l} f'_{x_1} = 0 \\ \vdots \\ f'_{x_n} = 0 \end{array}\right\}$ stationary pts
- Consider points $\underline{x}^*$ where $Df(\underline{x}^*)$ does not exist (critical, but not stationary)

} candidate points

- Check that all candidate points are in D°
- (In CONSTRAINED cases: check points in ∂D.)

SECOND ORDER CONDITIONS

(ALTERNATIVE OF ~~FMEA~~ EA 3.2.1./ ME 17.2)

If $\underline{x}^*$ is a stationary point for f, then

$H(f)(\underline{x}^*)$ pos. definite $\Rightarrow$ $\underline{x}^*$ local min

$H(f)(\underline{x}^*)$ neg. definite $\Rightarrow$ $\underline{x}^*$ local max

$H(f)(\underline{x}^*)$ indefinite with $\det H(f)(\underline{x}^*) \neq 0$ $\Rightarrow$ $\underline{x}^*$ saddlept (neither max/min)

(This is useful when f is not convex/not concave)

NOTE: if $\det H(f)(\underline{x}^*) = 0$, then closer examination is needed (Analogous to $f''(x^*) = 0$ in 1-var case).

CONCAVE / CONVEX OPTIMIZATION

When the function f is concave (convex), the situation is predictable (f must be C^1 in open ball around $\underline{x}^*$)

f concave $\Rightarrow$ $\arg\max\{f(x) \mid \underline{x} \in D\}$ is empty or <u>convex</u> set, any stationary pt f is a maximizer

f strictly concave $\Rightarrow$ $\arg\max\{f(\underline{x}) \mid \underline{x} \in D\}$ is empty or a pt, any stationary pt of f is a maximizer

f strictly quasiconcave $\Rightarrow$ $\arg\max\{f(\underline{x}) \mid \underline{x} \in D\}$ is empty or a pt, any local max is a global max

Alternative DEF: $f(\lambda \underline{x} + (1-\lambda)\underline{y}) > \min\{f(\underline{x}), f(\underline{y})\}$ for all $\lambda \in (0,1)$ and all $\underline{x}, \underline{y} \in D$.

quasi-concave: $\geqslant$ in the above def.