LECTURE 4

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DRE 7017

① CONVEX SETS

② SEPARATION THMS

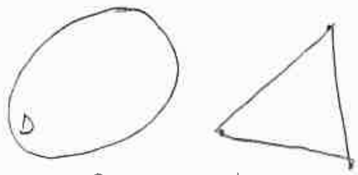
③ CONVEX / CONCAVE FUNCTIONS

FMEA 2.2-2.3, 2.5, 13.5-13.6
ME 21.1-21.2
(S 1.2, 1.6, 7.1-7.2, 8.1-8.3)

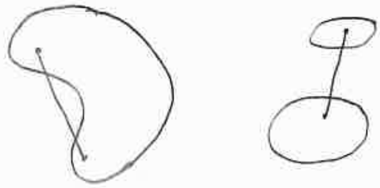
CONVEX SETS

Informally: "No fjords", "No islands"

In $\mathbb{R}^2$: D is a convex set if each pair of points in D can be joined by a line contained entirely in D.



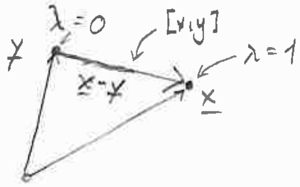
Convex sets



Not convex

In $\mathbb{R}^n$: A set D in $\mathbb{R}^n$ is called CONVEX if the line segment $[\underline{x},\underline{y}] = \{\lambda \underline{x} + (1-\lambda)\underline{y} \mid 0 \le \lambda \le 1\}$ is contained in D for all $\underline{x}$ and $\underline{y}$ in D.

RECALL: Line through $\underline{x}$ and $\underline{y}$

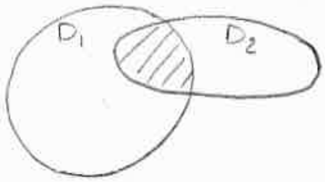


Line: $\underset{\text{point}}{\underline{y}} + \underset{\text{parameter}}{\lambda}(\underline{x}-\underline{y}) = \lambda\underline{x} + (1-\lambda)\underline{y}, \lambda \in \mathbb{R}$ direction

NOTE: • $\emptyset$ and $D = \{\underline{p}\}$ are convex

• $D_1, \dots, D_m \subseteq \mathbb{R}^n$ convex $\Rightarrow$ $D_1 \cap \dots \cap D_m$ convex

The union of convex sets is not convex in general.



• $D, E \subseteq \mathbb{R}^n$ convex $\Rightarrow$ $\underset{\text{vector sum}}{D+E} = \{\underline{x}+\underline{y} \mid \underline{x} \in D, \underline{y} \in E\}$ is convex

actually $aD + bE$ is convex for $a, b \in \mathbb{R}$.

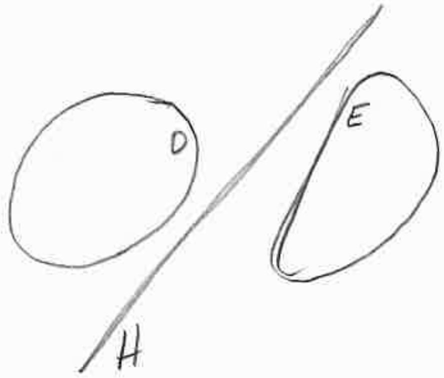
Proof: Let $\underline{z}, \underline{w} \in Q = aD + bE$, so $\underline{z} = a\underline{x}_1 + b\underline{y}_1$, $\underline{w} = a\underline{x}_2 + b\underline{y}_2$, $\underline{x}_i \in D$, $\underline{y}_i \in E$

Then $\lambda\underline{z} + (1-\lambda)\underline{w} = a(\underbrace{\lambda\underline{x}_1 + (1-\lambda)\underline{x}_2}_{\in D}) + b(\underbrace{\lambda\underline{y}_1 + (1-\lambda)\underline{y}_2}_{\in E})$

since D, E convex

$\in aD + bE = Q$.

SEPARATION THEOREMS



Idea:
Two disjoint sets in $\mathbb{R}^2$ can be separated by a line

$$H = \{\underline{x} \in \mathbb{R}^n \mid \underline{p} \cdot \underline{x} = a\} \subseteq \mathbb{R}^n$$
$$= \{\underline{x} \mid p_1 x_1 + \dots + p_n x_n = a\}$$

is called a HYPERPLANE when $\underline{p} \neq \underline{0} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.
(A $(n-1)$-dimensional linear supspace of $\mathbb{R}^n$)

FACT. Any hyperplane in $\mathbb{R}^n$ separates $\mathbb{R}^n$ in two convex half-spaces

$$H_+ = \{\underline{x} \mid \underline{p} \cdot \underline{x} \geq a\} \qquad H_- = \{\underline{x} \mid \underline{p} \cdot \underline{x} \leq a\}$$

Convex H_+: $\underline{x}_1, x_2 \in H_+$

$$\underline{p} \cdot (\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2) = \lambda(\underline{p} \cdot \underline{x}_1) + (1-\lambda)(\underline{p} \cdot \underline{x}_2)$$
$$\geq a\lambda + (1-\lambda) \cdot a$$
$$= a$$

THM (FMEA 13.6.3) SEPARATION THM
If $D, E \subseteq \mathbb{R}^n$ are two non-empty convex sets with $D \cap E = \emptyset$, then there exists a vector $\underline{p} \neq \underline{0}$ in $\mathbb{R}^n$ and a scalar a such that

$$\underline{p} \cdot \underline{x} \leq a \leq \underline{p} \cdot \underline{y} \quad \text{for all } \underline{x} \in D \text{ and } \underline{y} \in E.$$

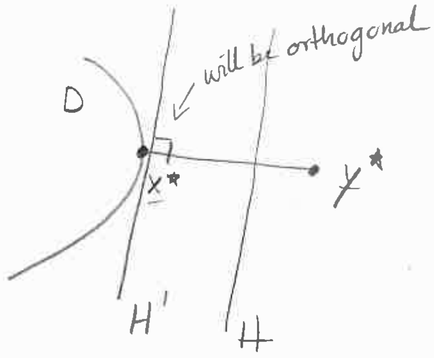
THM (FMEA 13.6.1) SPECIAL CASE FOR POINTS

$D \subseteq \mathbb{R}^n$ closed, convex set, $\underline{y}^* \in D^c$.

Then there exists a vector $\underline{p} \neq \underline{0}$ in $\mathbb{R}^n$ and $a \in \mathbb{R}$ such that

$$\underline{p} \cdot \underline{x} < a < \underline{p}\,\underline{y}^* \text{ for all } \underline{x} \in D.$$

EX.



Choose x^* s.t. $d(x^*, y^*)$ is minimal among all x^* in D.

$\underline{p} = \underline{y}^* - \underline{x}^*$

$H \overset{\circ}{=} \{ \underline{x} \mid \underline{p} \cdot \underline{x} = a$ with $\underline{p} \cdot \underline{x}^* < a < \underline{p} \cdot \underline{y}^* \}$

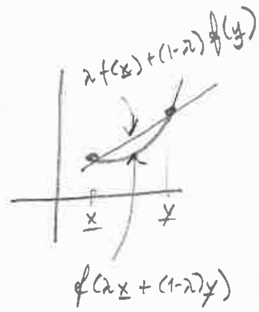
any a in this interval will do.

Called SUPPORTING HYPERPLANE (to D at x^*) when $a = \underline{p} \cdot \underline{x}^*$

CONVEX / CONCAVE FUNCTIONS

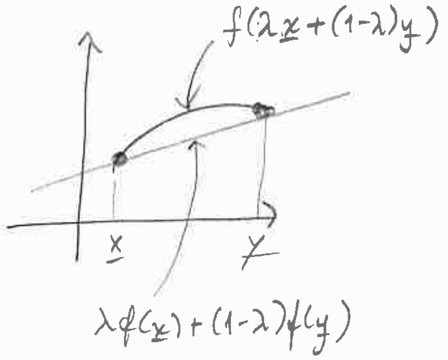
$D \subseteq \mathbb{R}^n \quad f: D \to \mathbb{R}$

DEF: • The function f is called CONVEX if
D is a convex set
and $f(\lambda \underline{x} + (1-\lambda)\underline{y}) \leq \lambda f(\underline{x}) + (1-\lambda) f(\underline{y})$
for all $\underline{x}$ and $\underline{y}$ in D. and all $\lambda \in [0,1]$.



• STRICT CONVEX if $<$

• CONCAVE if $\geq$
STRICT CONCAVE if $>$



• f concave $\Leftrightarrow -f$ convex

THM (FMEA 2.3.3)
Suppose that $f: D \to \mathbb{R}$ is a C^2-function defined on an open, convex set $D \subseteq \mathbb{R}^n$, then:
$H(f)(\underline{x})$ pos. semidefinite for all $\underline{x} \in D \Leftrightarrow f$ convex
$H(f)(\underline{x})$ neg. semidefinite for all $\underline{x} \in D \Leftrightarrow f$ concave

$H(f)(\underline{x})$ symmetric, so check this using
① Eigenvalues
or ② Principal minors

NOTE: $H(f)(\underline{x})$ pos. definite $\Rightarrow f(\underline{x})$ strictly convex
$H(f)(\underline{x})$ neg. definite $\Rightarrow f(\underline{x})$ strictly concave

QUASICONCAVE / QUASICONVEX FUNCTIONS

DEF: $f: D \to \mathbb{R}$, D convex set in $\mathbb{R}^n$

- The function f is QUASICONCAVE if the upper level set $U_f(a) = \{\underline{x} \in D \mid f(x) \geq a\}$ is a convex set for all a.
- The function f is QUASICONVEX if the lower level set $L_f(a) = \{\underline{x} \in D \mid f(x) \leq a\}$ is a convex set for all a. ($\Leftrightarrow$ If $-f$ is quasiconcave)

FACT: If $f(\underline{x})$ is concave, then $f(\underline{x})$ is quasiconcave
If $f(\underline{x})$ is convex, then $f(\underline{x})$ is quasiconvex
(But not conversely!)

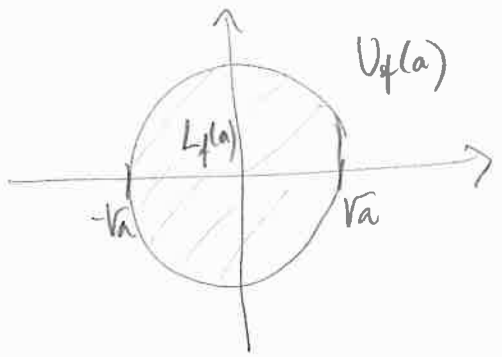
EX: $f(x,y) = x^2+y^2$ on $D = \mathbb{R}^2$

$H(f)(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is pos. def.

So f is strictly convex

$f'_x = 2x$ $\quad f''_{xx} = 2$
$f'_y = 2y$ $\quad f''_{xy} = 0$
$f''_{yy} = 2$

$U_f(a) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq a\}$
$L_f(a) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq a\}$



$L_f(a)$ is convex for all a, so f is quasiconvex.

$U_f(a)$ is not convex!

BORDERED HESSIAN CRITERION

THM (2.5.6 FMEA)

$f: D \to \mathbb{R}$ C^2, D open, convex set in $\mathbb{R}^n$.

Let
$$B_r(\underline{x}) = \begin{vmatrix} 0 & f_1'(\underline{x}) & \cdots & f_r'(\underline{x}) \\ f_1'(\underline{x}) & f_{11}''(\underline{x}) & & f_{1r}''(\underline{x}) \\ \vdots & \vdots & & \vdots \\ f_r'(\underline{x}) & f_{r1}''(\underline{x}) & \cdots & f_{rr}''(\underline{x}) \end{vmatrix}, \quad r = 1, \ldots, n.$$

↖ Bordered Hessian

Then ① A necessary condition for f to be quasiconcave is that $(-1)^r B_r(\underline{x}) \geq 0$ for all $\underline{x} \in D$ and all $r = 1, .., n$.

② A sufficient condition for f to be STRICTLY quasiconcave is that $(-1)^r B_r(\underline{x}) > 0$ for all $\underline{x}$ in D and all $r = 1, .., n$.

FACTS: ① Any strictly concave function is strictly quasiconcave

② A strictly quasiconcave function cannot have more than one global max.

EX: Consider $f(x,y) = x^2+y^2$, saw convex, try $B_r(\underline{x})$ on $-f$ to check

$$B_1(x,y) = (-1)\begin{vmatrix} 0 & -2x \\ -2x & 2 \end{vmatrix} = +4x^2 > 0$$

$$B_2(x,y) = (-1)^2\begin{vmatrix} 0 & -2x & -2y \\ -2x & -2 & 0 \\ -2y & 0 & -2 \end{vmatrix} = 2x\begin{vmatrix} -2x & 0 \\ -2y & -2 \end{vmatrix} - 2y\begin{vmatrix} -2x & -2 \\ -2y & 0 \end{vmatrix}$$

$$= 8x^2 + 8y^2$$

$$= 8(x^2+y^2) > 0$$

So $-f$ is strictly quasiconcave & has one max

Indeed f is strictly quasiconvex with one global minimum $(x,y) = (0,0)$...