# LECTURE 3

24.08.2020

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DRE 7017

① TOPOLOGY

② FUNCTIONS AND CONTINUITY

③ DERIVATIVES

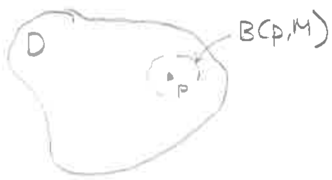
| | |
|---|---|
| FMEA | 2.9, 13.3 |
| ME | 13.4, 14 |
| S | 1.4, 3 |

# TOPOLOGY ON A METRIC SPACE $(X, d)$

Open ball in $X$: $B(p,M) = \{x \in X : d(x,p) < M\}$

Closed ball in $X$: $\overline{B}(p,M) = \{x \in X : d(x,p) \leq M\}$

DEF: • $D \subseteq X$ is <u>open</u> if for any $p \in D$ there is an open ball $B(p,M)$ such that $B(p,M) \subseteq D$

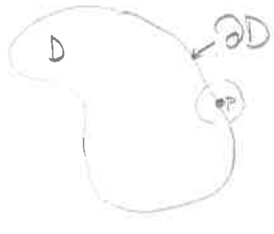




$(a,b)$

$a < x < b$

• $\partial D$ = the boundary points of $D$
= all points $p \in D$ s.t any open ball $B(p,M)$ contains points in both $D$ and $D^c$ (the complement)




$B(p,M) \cap D \neq \emptyset$

$B(p,M) \cap D^C \neq \emptyset$

for all $M > 0$

• $D^o = D \setminus \partial D$ are called interior points
= all points in $D$ that are not boundary pts.

NOTE: • $D \subseteq X$ is closed if $D^c$ is open

RESULT • $D \cap \partial D = \emptyset \iff D$ is open

• $\partial D \subseteq D \iff D$ is closed

DEF: A set $D \subseteq X$ is <u>BOUNDED</u> if there is a pt $p \in D$ and an open ball $B(p, M)$ s.t. $D \subseteq B(p, M)$.

DEF: $D \subseteq X$ is <u>COMPACT</u> if any sequence $\{x_i\}$ in $D$ has a subsequence $\{x_{i_k}\}$ that converges to a limit $x$ in $D$.

CONSEQUENCE: $D$ compact $\Rightarrow$ $D$ closed and bounded

For Euclidean space, the converse is also true:

THM (BOLZANO-WEIERSTRASS)

$D \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ $D$ is closed and bounded.

# FUNCTIONS AND CONTINUITY

$X = (X, d)$
$Y = (Y, d)$
metric spaces

**DEF:** A function $f: X \to Y$ is a rule that to each element in $X$ assigns one (and only one) value in $Y$

$$x \longmapsto f(x)$$

image of $x$ under $f$

$X$ domain
$Y$ codomain
$f(X)$ image of $f$

EX: ① $f: \mathbb{R} \to \mathbb{R}$
$x \longmapsto x^3 + 2x + 3$ polynomials in one variable

② $f: \mathbb{R}^2 \to \mathbb{R}^+$ — two variables
$(x,y) \longmapsto x^2 + y^2$
Domain $\mathbb{R}^2$
Image of $f = [0, \infty)$

③ $f: \mathbb{R} \to \mathbb{R}$
$x \longmapsto \frac{1}{x}$ Rational function
Domain: $\mathbb{R} \setminus \{0\}$ (not defined at $x = 0$.)
Image: $\mathbb{R} \setminus \{0\}$ (can never reach 0.)

④ $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \longmapsto e^{xy}$ exponential function

⑤ $f: D \to \mathbb{R}$ $D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$
$(x,y) \longmapsto \ln(x^2 + y^2)$ logarithmic function

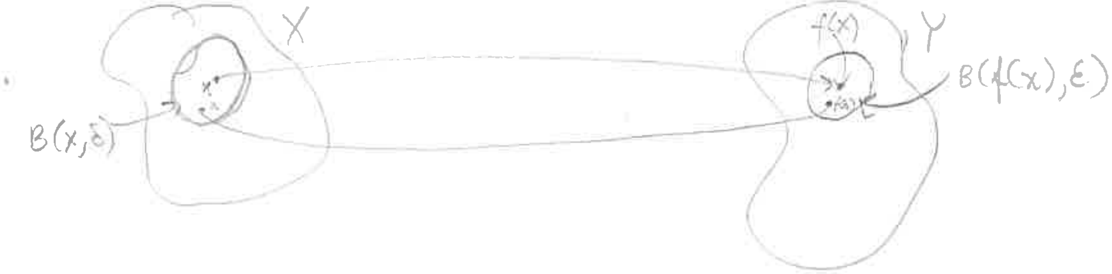
⑥ $T: \mathbb{R}^3 \to \mathbb{R}^3$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for a $3\times 3$-matrix $A$
operators / linear transformations

⑦ $J: C(I, \mathbb{R}) \to \mathbb{R}$
$f \longmapsto \int_0^1 f(x)\,dx$ functionals

DEF: Let $f: X \to Y$ be a function.
The function $f$ is called <u>CONTINUOUS AT $x \in X$</u> if for any $\varepsilon > 0$, there is a $\delta > 0$ s.t.
$f(a) \in B(f(x), \varepsilon)$ for all $a \in X$ with $a \in B(x, \delta)$

continuous at ONE pt

$(d(f(x), f(a)) < \varepsilon \longleftarrow d(x,a) < \delta)$




DEF continuous: $f: X \to Y$ is continuous if it is continuous at all points $x \in X$.

THM: If $f: X \to Y$ is continuous and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.
(The image of a compact set under a <u>cont. function</u> is compact.)

THM: If $D \subseteq \mathbb{R}^n$ is closed and bounded, and $f: D \to \mathbb{R}$ is continuous, then $f$ attains a max/min on $D$.

FACTS:
- All "elementary" functions on $\mathbb{R}^n$ are continuous on <u>their</u> domain (where they are defined...)
  (Polynomials, rational functions, exp, ln, trigonometric fn.
- Sums, differences, products, quotients and compositions preserve continuity

EX: ① $f(x) = \begin{cases} e^x & , x \geq 0 \\ 1-x & , x < 0 \end{cases}$ There COULD be a problem here for $x=0$...

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} e^x = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} 1-x = 1$$

② $C(I, \mathbb{R}) \longrightarrow \mathbb{R}$

$f \longmapsto \int_0^1 f(x)\,dx$ is continuous

③ $C(D, \mathbb{R}) = \{ f : D \to \mathbb{R} \mid f \text{ cont} \}$

metric space $\|f\|_{sup} = \sup_{x \in D} |f(x)|$

D compact $\Longrightarrow$ $C(D, \mathbb{R})$ complete

(any cauchy sequence converges)

Show continuous by $\varepsilon, \delta$-argument $\quad f(x) = \begin{cases} e^x, & x \geq 0 \\ 1-x, & x < 0 \end{cases}$

Given $\varepsilon > 0$ and consider for $x < 0$ ($x \geq 0$ ok)

$$|f(x) - f(0)| = |1 - x - e^0|$$
$$= |-x|$$
$$= |x - 0|$$

Choose $\delta = \varepsilon$, then for $|x-0| < \delta$ we have $|f(x) - f(0)| < \varepsilon$, so $f$ is continuous at $0$.

# DERIVATIVES

Alternative formulation $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$f: \mathbb{R} \to \mathbb{R}$ with derivative at $a$ $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (if it exists)

$f: \mathbb{R}^n \to \mathbb{R}$ partial derivative $f_i'(\underline{a}) = \lim_{h \to 0} \frac{f(\underline{a} + \varepsilon_i h) - f(\underline{a})}{h}$

$\varepsilon_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ith basis vector

$$f_i'(\underline{a}) = \frac{\partial f}{\partial x_i}(\underline{a})$$

TOTAL DERIVATIVE

$f: \mathbb{R}^n \to \mathbb{R}$ is said to be differentiable at $\underline{a}$ if there exists an $1 \times n$-matrix $A$ such that

$$\lim_{\underline{h} \to 0} \frac{\| f(\underline{a} + \underline{h}) - f(\underline{h}) - A h \|}{\| \underline{h} \|} = 0$$

If $A$ exists, it is called the TOTAL DERIVATIVE of $f$ at $\underline{a}$.

REMARK ① If $f$ has a derivative at $\underline{a}$, it is unique, and $f'(\underline{a}) = Df(\underline{a}) = \left( \frac{\partial f}{\partial x_1}(\underline{a}) \quad \frac{\partial f}{\partial x_n}(\underline{a}) \right)$ Jacobian matrix at $\underline{a}$

$= \nabla f(\underline{a})$ $(\mathbb{R}^n \to \mathbb{R})$ Gradient vector at $\underline{a}$

② If all $\frac{\partial f}{\partial x_i}$ exist and are continuous around $\underline{a}$, then $f$ is differentiable at $\underline{a}$ (FMEA p. 97)

③ Note that the total derivative of $f$ at a point $\underline{a}$ is a $1 \times n$-matrix. We are sometimes interested in the derivative of $f$ at $\underline{a}$ along a vector $\underline{x}$

$$f_{\underline{x}}'(\underline{a}) = f'(\underline{a}) \cdot \underline{x}$$

DEF:

$C^1$-functions: All partial derivatives exist and are continuous

$C^2$-functions: All second order partial derivatives exist and are continuous.

$C^2 \Rightarrow C^1 \Rightarrow C$ = continuous

THM: If $f: \underset{\mathbb{R}^n}{\overset{}{D}} \subseteq \mathbb{R}^n$, $f: D \longrightarrow \mathbb{R}$ is $C^2$, then

$$D^2 f = H(f) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ \vdots & & & \vdots \\ f''_{n1} & \cdots & \cdots & f''_{nn} \end{pmatrix}$$ The Hessian matrix

is a symmetric matrix (due to Young's thm) $C^2$ is required