LECTURE 2

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DRE 7017

1. SPACES
2. SEQUENCES

(ES 1-2)
FMEA A.1 - A.3, 13.1 - 13.2
ME A1, 10, 12, 29

EUCLIDEAN SPACE AND GENERAL VECTOR SPACES

VECTOR SPACES

Set V of elements called vectors
and operations

- Addition : $V \times V \xrightarrow{+} V$
  $(\underline{v}, \underline{w}) \longmapsto \underline{v} + \underline{w}$
- Scalar mult. $\mathbb{R} \times V \longrightarrow V$ $\mathbb{R}$ = real numbers
  $(r, \underline{v}) \longrightarrow r \cdot \underline{v}$

such that for all $\underline{u}, \underline{v}, \underline{w} \in V$ and $r, s \in \mathbb{R}$

1. $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ Commutativity (+)
2. $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ Associativity (+)
3. $\underline{v} + \underline{0} = \underline{v}$ Additive identity element
4. $\underline{v} + (-\underline{v}) = \underline{0}$ Additive inverse
5. $(rs) \cdot \underline{v} = r(s\underline{v})$ Compatibility
6. $r(\underline{u} + \underline{v}) = r\underline{u} + r\underline{w}$ Distributivity $(\cdot)$ wrt $V_{(+)}$
7. $(r+s)\underline{v} = r\underline{v} + s\underline{v}$ Distributivity $(\cdot)$ wrt. $\mathbb{R}_{(+)}$
8. $1 \cdot \underline{v} = \underline{v}$ Multiplicative identity elem

EUCLIDEAN SPACE (canonical example of vector space)

$V = \mathbb{R}^n = \{\underline{v} : \underline{v} \text{ is } n\text{-vector}\}$

$= \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} : v_1, v_2, \ldots, v_n \in \mathbb{R} \right\}$

$$\underline{v} + \underline{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} \left(= \begin{pmatrix} w_1 + v_1 \\ \vdots \\ w_n + v_n \end{pmatrix} \overset{1}{=} \underline{w} + \underline{v} \right)$$

$$r \cdot \underline{v} = r \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

$$\left(\begin{array}{l} \underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \ 3 \\ -\underline{v} = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} \ 4 \end{array} \right)$$

Check the other properties

SPACE OF CONTINUOUS FUNCTIONS (2nd example)

$V = C(I,\mathbb{R})$ where $I = [0,1] \subseteq \mathbb{R}$

$= \{ f : I \to \mathbb{R} \mid f$ a $\overbrace{\text{continuous}}^{\text{real}}$ function on $I \}$

$+$ Usual addition $(f+g)(x) = f(x) + g(x)$

$\cdot$ Scalar multiplication. $(rf)(x) = r \cdot f(x)$

Ex: e^x, x^2, $\sqrt{x}$

INNER PRODUCT SPACES

An <u>inner product</u> on a vector space V is a product

$$V \times V \longrightarrow \mathbb{R}$$

$$(\underline{v}, \underline{w}) \longmapsto \langle \underline{v}, \underline{w} \rangle = \underline{v} \cdot \underline{w}$$

such that for $\underline{u}, \underline{v}, \underline{w} \in V$ and $r, s \in \mathbb{R}$

1. $\langle \underline{v}, \underline{w} \rangle = \langle \underline{w}, \underline{v} \rangle$ — Symmetry
2. $\begin{cases} \langle r\underline{v}, \underline{w} \rangle = r\langle \underline{v}, \underline{w} \rangle \\ \langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle \end{cases}$ — Linearity

   (Combined: $\langle r\underline{u} + s\underline{v}, \underline{w} \rangle = r\langle \underline{u}, \underline{w} \rangle + s\langle \underline{v}, \underline{w} \rangle$)
3. $\langle \underline{v}, \underline{v} \rangle \geqslant 0$ and $\langle \underline{v}, \underline{v} \rangle = 0 \iff \underline{v} = \underline{0}$

EX: EUCLIDEAN SPACE $\mathbb{R}^n$

$$\left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = v_1 \cdot w_1 + \dots + v_n \cdot w_n$$

(the usual dot product) $\underline{v} \cdot \underline{w}$

THM:
CAUCHY-SCHWARZ INEQUALITY

If V is an inner product space, then

$$|\langle \underline{v}, \underline{w} \rangle| \leq \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}} \cdot \langle \underline{w}, \underline{w} \rangle^{\frac{1}{2}}$$

for all $\underline{v}, \underline{w} \in V$, and equality holds iff $\{\underline{v}, \underline{w}\}$ are linearly indep.

PROOF:
- Assume $\underline{w} \neq \underline{0}$. ($\underline{w} = \underline{0}$ is trivial since $\langle \underline{v}, \underline{w} \rangle = 0$)
- Construct a vector $\underline{u} = \underline{v} - \dfrac{\langle \underline{v}, \underline{w} \rangle}{\langle \underline{w}, \underline{w} \rangle} \cdot \underline{w}$
- Compute $\langle \underline{u}, \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle - \dfrac{\langle \underline{v}, \underline{w} \rangle}{\langle \underline{w}, \underline{w} \rangle} \langle \underline{w}, \underline{w} \rangle$

  $= 0$
- Let $r = \dfrac{\langle \underline{v}, \underline{w} \rangle}{\langle \underline{w}, \underline{w} \rangle}$ and write $\underline{v} = \underline{u} + r\underline{w}$
- $\langle \underline{v}, \underline{v} \rangle = \langle \underline{u} + r\underline{w}, \underline{u} + r\underline{w} \rangle$

  $= \langle \underline{u}, \underline{u} \rangle + 2r\langle \underline{u}, \underline{w} \rangle + r^2 \langle \underline{w}, \underline{w} \rangle$

$$= \underbrace{\langle \underline{u}, \underline{u} \rangle}_{\geq 0} + r^2 \langle \underline{w}, \underline{w} \rangle$$

$$\geq r^2 \langle \underline{w}, \underline{w} \rangle$$

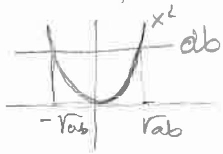
$$= \frac{\langle \underline{v}, \underline{w} \rangle^2}{\langle \underline{w}, \underline{w} \rangle^{\cancel{2}}} \cancel{\langle \underline{w}, \underline{w} \rangle}$$

Altogether

$$\langle \underline{v}, \underline{v} \rangle \geq \frac{\langle \underline{v}, \underline{w} \rangle^2}{\langle \underline{w}, \underline{w} \rangle}$$

So: $\langle \underline{v}, \underline{w} \rangle^2 \leq \langle \underline{v}, \underline{v} \rangle \cdot \langle \underline{w}, \underline{w} \rangle$

$x^2 \leq a \cdot b$ for $a, b \geq 0$



$$\underline{|\langle \underline{v}, \underline{w} \rangle| \leq \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}} \langle \underline{w}, \underline{w} \rangle^{\frac{1}{2}}}$$

$-\sqrt{ab} \leq x \leq \sqrt{ab}$

$|x| \leq \sqrt{ab}$

abs. value

- Equality holds $\Leftrightarrow$ $\langle \underline{u}, \underline{u} \rangle = 0$

  so $\underline{u} = 0$

  $\parallel$

  $\underline{v} - r \cdot \underline{w}$,

  then $\underline{v} = r \cdot \underline{w}$, so linearly dependent.

$\square$

PROPERTIES OF INNER PRODUCT SPACES

Length of $\underline{v}$: $\|\underline{v}\| = \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}}$ where $\|\underline{v}\| \geq 0$

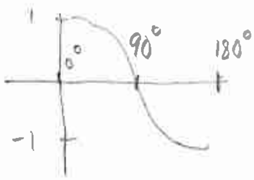
and $\|\underline{v}\| = 0 \Leftrightarrow \underline{v} = \underline{0}$

Angle between $\underline{v}$ and $\underline{w}$: $\alpha \in [0°, 180°]$ (unique)

$$\cos \alpha = \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{v}\| \cdot \|\underline{w}\|}$$

(Measures deviation from equality in C.S.)

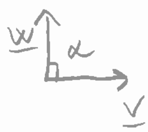
$(|\cdot| \leq 1$ by C.S.$)$



$\alpha = 0° \Rightarrow \underline{v} = r \cdot \underline{w}$ for $r > 0$

$\alpha = 180° \Rightarrow \underline{v} = r \cdot \underline{w}$ for $r < 0$

$\alpha = 90° \Leftrightarrow \langle \underline{v}, \underline{w} \rangle = 0$



$\underline{v} \perp \underline{w}$

perpendicular

NORMED VECTOR SPACE

A normed vector space is a vector space V with a norm function ("length")

$$V \to \mathbb{R}$$
$$\underline{w} \mapsto \|\underline{w}\|$$

such that for $\underline{v}, \underline{w} \in V$ and $r \in \mathbb{R}$

Nonnegative / Positive on nonzero vectors ① $\|\underline{w}\| \geq 0$ for all $\underline{w} \in V$ and $\|\underline{w}\| = 0$ only if $\underline{w} = \underline{0}$

Scalar ② $\|r\underline{w}\| = |r| \cdot \|\underline{w}\|$ ← abs. value

$\triangle$ inequality ③ $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$

$\langle \underline{v}, \underline{v} \rangle \implies \|\underline{v}\| = \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}}$

REMARK: An inner product induces a norm
all inner product spaces are normed spaces.

EX. Euclidean norm: $\underline{v} \in \mathbb{R}^n = V$

$$\|\underline{v}\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{\frac{1}{2}} = (\underline{v} \cdot \underline{v})^{\frac{1}{2}} = \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}}$$

$n = 2$ is Pythagoras' theorem.

EX. Sup norm: $V = C(I, \mathbb{R})$

$$\|f\|_{\sup} = \sup \{ f(x) : x \in I \}$$
$$= \sup_{x \in I} f(x)$$

Need: $J \subseteq \mathbb{R}$ a set. An UPPER BOUND for J is M such that $M \geq j$ for any $j \in J$.

"Supremum" $\sup J \overset{\text{def}}{:=}$ least upper bound for J

"Infimum" $\inf J :=$ greatest lower bound for J.

$J = \{ 1/n : n \in \mathbb{N} \} = \{ 1, \frac{1}{2}, \frac{1}{3}, \dots \}$ $\sup J = 1 (\in J)$ $\inf J = 0 \notin J)$

METRIC SPACES (X, d) Can be a vector space V!

A metric space is a set X together with a metric (distance function) $d: X \times X \to \mathbb{R}$ such that for $x, y, z \in X$

1. $d(x,y) = 0 \iff x = y$ identity of indiscernibles
2. $d(x,y) = d(y,x)$ symmetry
3. $d(x,z) \leq d(x,y) + d(y,z)$ triangle ineq.

and it follows that

$$d(x,y) \geq 0$$

$$\underbrace{d(x,x)}_{=0} \leq d(x,y) + d(y,x) = d(x,y) + d(x,y) = 2d(x,y)$$

Then divide by 2.

REMARK:

An inner product $\langle \underline{v}, \underline{v} \rangle$ induces a norm $\|\underline{v}\| = \langle \underline{v}, \underline{v} \rangle^{\frac{1}{2}}$ which induces a metric (distance fn) $d(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\|$

EX: Euclidean metric

$$d(\underline{v}, \underline{w}) = \|\underline{v} - \underline{w}\| = \sqrt{(v_1 - w_1)^2 + \cdots + (v_1 - w_n)^2}$$

SEQUENCES

(X, d) metric space

A sequence is a collection of pts $x_i \in X$, $i \in \mathbb{N} = \{1, 2, 3, \dots\}$

(x_i) or $\{x_i\}$

<u>DEF:</u> The sequence $\{x_i\}$ converges to a limit $x \in X$

$$\{x_i\} \to x$$

$$\lim_{i \to \infty} x_i = x$$

if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_i, x) < \varepsilon$ for $i > N$

<u>Informally:</u> No matter how close to x we go, it is always possible to find a position in the sequence, such that all elements from then on are closer.

EX: $V = \mathbb{R}$ $\quad d(x,y) = |x-y|$ abs. value.

$x_i = \frac{1}{2i}$ $\qquad \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$ $\qquad$ Want to show $\{x_i\} \to 0$

Given $\varepsilon > 0$, consider $d(x_i, 0) = |x_i - 0| = |x_i| = \frac{1}{2i}$

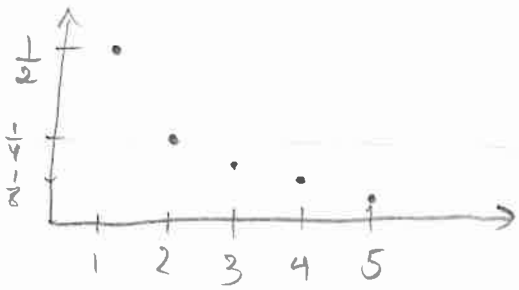
We want $\frac{1}{2i} < \varepsilon \iff \frac{1}{2\varepsilon} < i$

So choose $N = \frac{1}{2\varepsilon} \iff \varepsilon = \frac{1}{2N}$

Then $|x_i| = \frac{1}{2i} < \frac{1}{2N} = \varepsilon$ for $i > N$,

(Bigger denominator $\Rightarrow$ smaller number)

and the sequence converges



if $\varepsilon = \frac{1}{4}$, then $N = 2$ is enough

CAUCHY SEQUENCE

If (X,d) is a metric space, then a sequence $\{x_i\}$ in X is called a Cauchy sequence if:

For each $\varepsilon > 0$, there exists N such that for $i,j > N$ then $d(x_i, x_j) < \varepsilon$

Informally: A sequence whose elements become arbitrarily close to each other from some element on, i.e. all but finitely many are less than that given distance apart.

Purpose: Easier to check Cauchy-criterion than limit.

RESULTS:

① If $\{x_i\} \to x$, then $\{x_i\}$ is Cauchy.

Proof: Given $\varepsilon > 0$, since $\{x_i\} \to x$, there exists N s.t. $d(x_i, x) < \frac{\varepsilon}{2}$ for $i > N$;

Then Δ-ineq:

$$d(x_i, x_j) \le d(x_i, x) + d(x_j, x) < \varepsilon$$

for $i,j > N$.

The converse is not always true, BUT it holds for $(\mathbb{R}^n, d)$

② If $\{\underline{x}_i\}$ is a Cauchy sequence in $\mathbb{R}^n$, then $\{\underline{x}_i\} \to \underline{x}$ for some $\underline{x} \in \mathbb{R}^n$

DEF: In general metric space (X,d) is called COMPLETE if any Cauchy sequence in X converges to some $x \in X$.

② $(\mathbb{R}^n, d)$ is complete.

③ If $\{x_i\}$ converges, it is bounded, i.e., there is an open ball $B(\underline{p}, R) = \{\underline{x} \in X \,|\, d(\underline{x}, \underline{p}) < R\}$ that contains $\{x_i : i\}$

SUBSEQUENCE

A subsequence of $\{x_i\}$ is a sequence obtained by picking infinitely many elements from $\{x_i\}$

$$x_{j_1}, x_{j_2}, \dots, x_{j_k}, \dots \qquad j_1 < j_2 < \dots < j_k < \dots \in \mathbb{N}$$

If $\{x_i\} \to x$, any subseq. will converge to x as well.
But not conversely! EX: $\underset{\text{does not converge}}{\{(-1)^i\}}$ with subseq. $\underset{1}{\{(-1)^{2i}\}}$