LECTURE 1

17.08.2020

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DRE 7017

1. MATRICES AND LINEAR SYSTEMS
2. QUADRATIC FORMS AND DEFINITENESS

FMEA	1.1 - 1.7
ME	6 - 9, 23
(GE	1 - 3)

① MATRICES AND LINEAR SYSTEMS

MATRICES:

An $\underset{\text{rows}}{m} \times \underset{\text{colums}}{n}$ - matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}), \quad a_{ij} \in \mathbb{R}$$

A n-vector $\underline{v}$
column-vector

$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

OPERATIONS

- Addition / subtraction
  A, B same size

$$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$
$$A - B = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij})$$

- Scalar multiplication

$$c \cdot A = c \cdot (a_{ij}) = (c\, a_{ij})$$
c scalar (number)

- Multiplication
  # col's in A
  = # rows in B
  In general: $BA \neq AB$,
  BA may not even exist!

$$\underset{m \times n}{A} \cdot \underset{n \times p}{B} = (a_{ij}) \cdot (b_{ij}) = (c_{ij}) = \underset{m \times p}{C}$$

where $c_{ij} = a_{i1} \cdot b_{1j} + \ldots + a_{in} \cdot b_{nj}$
(dot product between row i in A and col j in B)

- Transpose

$$\underset{m \times n}{A} \leadsto \underset{n \times m}{A^T} \quad \text{(other notation } A^t, A^*\text{)}$$

$$A = (a_{ij}), \qquad A^T = (a_{ji})$$

$$(AB)^T = B^T A^T$$

SPECIAL MATRICES

Zero matrix $O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ $A + O = O + A = A$

Identity matrix $I = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & & \\ 0 & \cdots & \cdots & & 1 \end{pmatrix}$ $A \cdot I = I \cdot A = A$ (multiplicative unit)

Square matrix A $n \times n$ $\#\text{rows} = \#\text{cols}$

Symmetric matrix $A^T = A$

Diagonal matrix $D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & & \\ & & \ddots & & 0 \\ 0 & \cdots & \cdots & 0 & d_3 \end{pmatrix}$

Upper/lower triangular matrix $A = \begin{pmatrix} d_1 & * & \cdots & * \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & d_n \end{pmatrix}$ Upper

Note: If A and B are triangular matrices of the same type (upper/lower - or diagonal), then so is $AB = (a_{ij})(b_{ij}) = (c_{ij})$ with $c_{ii} = a_{ii} \cdot b_{ii}$.

Inverse matrix An inverse matrix of A is a matrix A^{-1} s.t. $A \cdot A^{-1} = I = A^{-1}A$

If it exists, it is unique.

Note: $(AB)^{-1} = B^{-1}A^{-1}$ because $(AB)^{-1}AB = I$

by uniqueness $B^{-1}\underbrace{A^{-1}A}_{I}B = I$

$B^{-1}B = I$

DETERMINANTS

$$\underset{n\times n}{A} \longrightarrow \det(A) = |A|$$
(a number)

$n=1$ $A = (a)$ $|A| = a$

$n=2$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$n=3$ $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + C \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

n $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} C_{1n}$$ ← cofactor expansion on 1st row.

where $C_{ij} = (-1)^{i+j} M_{ij}$
cofactor

M_{ij} = determinant of submatrix where row i and col j are deleted.
minor

$$|AB| = |A| \cdot |B|$$

$$|A^T| = |A|$$

Link to inverse matrices: A^{-1} exists $\Leftrightarrow |A| \neq 0$

$$A^{-1} = \frac{1}{|A|} \underset{\text{adjoint}}{\operatorname{adj}(A)} = \frac{1}{|A|} (C_{ij})^T$$

$n=2$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $|A| = ad - bc$ $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

LINEAR SYSTEMS (OF EQUATIONS)

m linear equations in n variables

$m \times n$ linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad \underset{n\times 1}{\underline{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \underset{m\times 1}{\underline{b}} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

On matrix form

$$A\underline{x} = \underline{b}$$ with unique solution $\underline{x} = A^{-1}\underline{b}$ if $m = n$ and $|A| \neq 0$

Augmented matrix

$$(A\,|\,b) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

Gaussian elimination:
General method for solving linear systems using augm. matrix.

- Elementary row operations
  1. Switch two rows $\updownarrow$
  2. Multiply a row with a constant ($\neq 0$) $c \cdot R_i$
  3. Add a multiple of one row to another $cR_i + R_j$
- GOAL: Echelon form (can always be achieved, not uniquely)
  1. All zero rows are below other rows
  2. All entries under a $\underbrace{\text{first non-zero entry}}_{\text{pivot}}$ in a row must be zero
- From echelon form, backwards substitution solves the system

(8.20 a) HE p. 172)

EXAMPLE $\left.\begin{array}{l} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{array}\right\}$ $\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 1 & 3 \end{array}\right)$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right) -2R_1 \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -1 \end{array}\right) \cdot -1R_2$$

$$\sim \left(\begin{array}{cc|c} \underline{1} & 1 & 3 \\ 0 & \underline{1} & 1 \end{array}\right)$$

Echelon form:

1. $x_1 + x_2 = 3$, $\underline{x_2 = 1}$. $x_1 + 1 = 3$, $\underline{x_1 = 2}$

$R_1 - 1 \cdot R_2$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

Reduced echelon form

2. Translates to $\underline{\underline{x_1 = 2}}$, $\underline{\underline{x_2 = 1}}$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \det A = 2 \cdot 1 - 1 \cdot 1 = 1$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

3. $$\underline{x} = A^{-1} \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$$

Reduced echelon form:

① All pivots are 1

② All entries over a pivot are 0. } Unique

Solutions

THM: Any linear system has .

① One solution $\Leftrightarrow$ Pivot positions in all but the last col.

② No solution $\Leftrightarrow$ Pivot position in the last col.

③ Infinitely many solutions $\Leftrightarrow$ Columns without pivots (except last col.)
free variables.

Rank: Rank of A = # pivot positions in A $n \times m$-matrix

$Nul(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$ $\quad A\underline{x} = \underline{0}$ homogeneous linear system

Nullspace

$$\dim Nul(A) = n - rk(A)$$

EX.
$$\left.\begin{aligned} x_1 + 3x_2 - 2x_3 + 3x_4 &= 0 \\ 2x_3 + x_4 &= 0 \end{aligned}\right\}$$

$$\left(\begin{array}{cccc|c} \underline{1} & 3 & -2 & 3 & 0 \\ 0 & 0 & \underline{2} & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} \underline{1} & 3 & 0 & 4 & 0 \\ 0 & 0 & \underline{1} & \frac{1}{2} & 0 \end{array}\right)$$

Pivots

s free vars. t , $s,t \in \mathbb{R}$

$x_1 = -3s - 4t$

$x_3 = -\frac{1}{2} t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s - 4t \\ s \\ -\frac{1}{2}t \\ t \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4 \\ 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} t$$

$s,t \in \mathbb{R}$

$\dim Nul\, A = 2$, parametrized by s and t.

LINEAR INDEPENDENCE

set $\rightarrow$ $B = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ $\rightsquigarrow$ $A = (\underline{v}_1 \mid \underline{v}_2 \mid \dots \mid \underline{v}_n)$

n m-vectors $\qquad$ $m \times n$-matrix

B is linearly independent if

$x_1\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n = \underline{0}$ has only $\underline{x} = \underline{0}$ as solution

$\leftrightarrow \operatorname{rk} A = n$ (all pivots)

otherwise B is linearly dependent.

$\operatorname{rk} A$ = max number of lin. indep. vectors among B

= maximal order of a non-zero minor in A

$\underset{n \times n}{A}$ $\qquad \operatorname{rk} A = n \Leftrightarrow \det(A) \neq 0.$

EIGENVALUES AND EIGENVECTORS

A $n \times n$ - matrix

DEF: $\underline{v} \neq \underline{0}$ is an eigenvector with eigenvalue λ for A if there exists a number λ s.t.

$$A\underline{v} = \lambda \underline{v}.$$

NOT: $E_\lambda = \{\underline{v} : A\underline{v} = \lambda \underline{v}\}$ ← set of eigenvectors corresp. to λ.

COMP:

$$A\underline{v} = \lambda \underline{v}$$

$$A\underline{v} - \lambda \underline{v} = 0$$

$(A - \lambda I)\underline{v} = 0$ homogeneous system, so $\underline{v} = \underline{0}$ is a sol.

other solutions $\Leftrightarrow$ $\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$

① Find eigenvalues by solving deg n-pol $\det(A - \lambda I) = 0$

$\{\lambda_1, \dots, \lambda_r\}$, $r \leq n$ $m_i = \text{mult } \lambda_i$ = multiplicity of λ_i as solution of char. pol

$\Leftrightarrow (\lambda - \lambda_i)^{m_i}$ is factor

② Corresp. eigenvectors $E_{\lambda_i} = \text{Nul}(A - \lambda_i I)$

$$1 \leq \dim E_{\lambda_i} \leq \text{mult } \lambda_i$$

DIAGONALIZATON

DEF: A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D

such that $P^{-1}AP = D \iff A = PDP^{-1}$

Application: If A is diagonalizable, then

$$A^n = \underbrace{(PD\cancel{P^{-1}})\cdot(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})}_{n}$$

$$= PD^nP^{-1}$$

Fact • A is diagonalizable $\iff$ ① n eigenvalues (w/ mult) $\sum m_i = n$

② $\dim E_{\lambda_i} = m_i$

• Then $P = (\underline{v}_1 \mid \underline{v}_2 \mid \dots \mid \underline{v}_n)$ $\quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

linearly indep eigenvectors

Facts: ① A symmetric $\Rightarrow$ A diagonalizable

② If A is diagonalizable:

$\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A$

$\lambda_1 + \lambda_2 + \dots + \lambda_n = \operatorname{tr} A$

$= a_{11} + a_{22} + \dots + a_{nn}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

QUADRATIC FORMS & DEFINITENESS

$\propto$

Polynomial of degree 2 in n variables x_i

$$Q(x_1, \dots, x_n) = c_{11}x_1^2 + c_{12}x_1x_2 + \dots + c_{nn}x_n^2$$

quadratic form

$$= \underline{x}^T \cdot A \cdot \underline{x} \qquad A = \begin{pmatrix} c_{11} & \frac{1}{2}c_{12} & \cdots & \frac{1}{2}c_{1n} \\ \frac{1}{2}c_{12} & c_{22} & & \vdots \\ \frac{1}{2}c_{1n} & \cdots & \cdots & c_{nn} \end{pmatrix}$$

symmetric matrix

c_{ij} is the coefficient of x_ix_j

EX (16.1 d)

$$(x_1, x_2, x_3)\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -x_1^2 + 2x_1x_2 - x_2^2 + 2x_3^2$$

DEF:

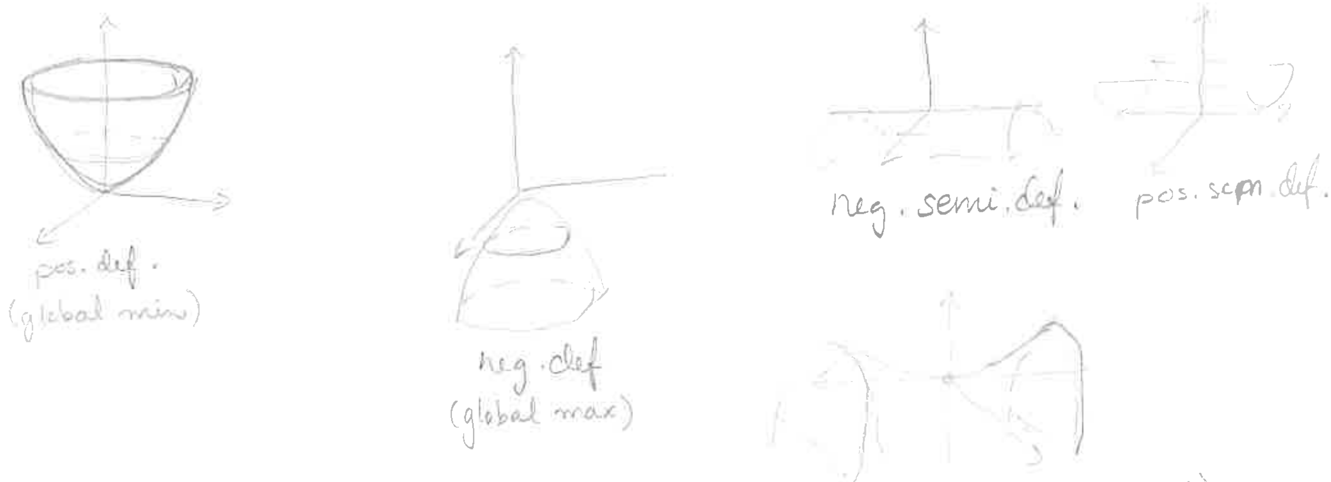
$Q(\underline{x}) \geq 0 \quad \forall\, \underline{x}$: Q positive semidefinite

$Q(\underline{x}) \leq 0$ — " — : Q negative semidefinite

neither : Q indefinite

$Q(x) > 0$ for all $\underline{x} \neq \underline{0}$: Q pos. def.

$Q(x) < 0$ — " — : Q neg. def.



pos. def. (global min)

neg. def (global max)

neg. semi. def.

pos. sem. def.

indefinite (saddle pt)

REMARK: Any $n \times n$-matrix corresponds to a quadratic form.
A quadratic form corresponds to a symmetric $n \times n$-matrix, say A.

EX: $Q(\underline{x}) = (x_1\ x_2)\begin{pmatrix} 2 & -3 \\ 7 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 2x_1 - 3x_2 \\ 7x_1 - 8x_2 \end{pmatrix} = 2x_1^2 - 3x_1x_2 + 7x_1x_2 - 8x_2^2$

$= 2x_1^2 + 4x_1x_2 - 8x_2^2$

$= (x_1\ x_2)\begin{pmatrix} 2 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

RESULT 1. A $n \times n$ symmetric matrix

A pos. semi. def. $\iff \lambda_1, \lambda_2, \dots, \lambda_n \geq 0$

pos defn $\lambda_1, \lambda_2, \dots, \lambda_n > 0$

A neg semidef $\iff \lambda_1, \lambda_2, \dots, \lambda_n \leq 0$

neg def $\lambda_1, \dots, \lambda_n < 0$

A indefinite $\iff$ Both positive & negative eigenvalues.

RESULT 2. A $n \times n$ symmetric matrix

A pos def $\iff D_1, D_2, \dots, D_n > 0$

A neg. def $\iff D_1 < 0, D_2 > 0, D_3 < 0, \dots, (-1)^n D_n > 0$

A pos. semidef $\iff \Delta_1, \Delta_2, \dots, \Delta_n \geq 0$

A neg semidef $\iff \Delta_1 \leq 0, \Delta_2 \geq 0, \dots, (-1)^n \Delta_n \geq 0$

D_i and Δ_i for a general 3×3-matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

D_i: leading principal minor of order i

$D_1 = |a_{11}|$

$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det A$

$$\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}$$

Δ_i principal minors of order i; $\binom{n}{i}$ for each $1 \leq i \leq n$

In total: $\sum_{i=1}^{n} \binom{n}{i} - 1$

Δ_1: ① $|a_{11}|$ ② $|a_{22}|$ ③ $|a_{33}|$

Δ_2: ① $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ ② $\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ ③ $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Δ_3: det A

Delete $(n-k)$ rows & $(n-k)$ the same columns in all possible combinations

EX: $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$D_1 = -1$

$D_2 = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = \underline{0}$

$D_3 = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -1 \cdot \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix}$

$= -2 + 2 = \underline{0}$

So leading principal matrices are not enough!

Δ_1: $-1 \quad -1 \quad -2 \quad < 0$

Δ_2: $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0 \quad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 \quad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 \quad \geqslant 0$

$\Delta_3 = D_3 = 0 \leq 0$

So A is negative semidefinite.

Eigenvalues & eigenvectors

$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & -2-\lambda \end{vmatrix}$

$= -(1+\lambda)^2(2+\lambda) + (2+\lambda)$

$= (2+\lambda)(1 - 1 - 2\lambda - \lambda^2)$

$= -(2+\lambda) \cdot \lambda \cdot (2+\lambda)$

$= -\lambda(\lambda+2)^2$

$\lambda_1 = 0 \ (m_1 = 1) \quad \lambda_2 = -2 \quad (m_2 = 2)$

$\underline{v}_1$: $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \underline{1} & -1 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & \underset{\downarrow}{0} & 0 & 0 \end{array}\right) \quad \underline{v}_1 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot t, \ t \in \mathbb{R}$

$$\underline{v}_2: \begin{pmatrix} -1+2 & 1 & 0 & 0 \\ 1 & -1+2 & 0 & 0 \\ 0 & 0 & -2+2 & 0 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} \quad & \downarrow & \downarrow & \quad \\ & s & t & \end{matrix} \end{array}$$

$$E_2 = \left\{ \begin{pmatrix} -s \\ s \\ t \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \quad , s,t \in \mathbb{R} \right\}$$

$$\left. \begin{aligned} \lambda_1 \cdot \lambda_2^2 &= 0 \cdot (-2)^2 = 0 = \det A \\ \lambda_1 + 2\lambda_2 &= -4 = trA. \end{aligned} \right\}$$ But not sufficient (a priori) to determine $\lambda_1, \lambda_2, \lambda_3$ (unless $n=2$)

$$\underline{v}_2^1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{v}_2^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

REDUCED RANK CRITERION (RCC - Eriksen 2017)

If A is a symmetric $n \times n$-matrix of $rk\, A = r < n$, then

$D_1, D_2, \dots, D_r > 0 \Rightarrow A$ pos. semidef.

$D_1 < 0, D_2 > 0, \dots, (-1)^r D_r > 0 \Rightarrow A$ neg. semidef.

See example.

DYNAMICAL SYSTEMS / MARKOV PROCESS

$\underline{u}_t$ state n-vector at time t

$\underline{u}_{t+1} = A \underline{u}_t$ A $n \times n$-transition matrix

$\underline{u}_0$ initial state

Goal: Write $\underline{u}_t$ as a function of $\underline{u}_0$ to evaluate $\lim_{t \to \infty} \underline{u}_t$

Method: ① Find eigenvalues and corresponding eigenvectors of A

② Write $\underline{u}_0$ as a linear comb of eigenvectors

$$\underline{u}_0 = \sum_{i=1}^{n} s_i \underline{v}_i \quad (\text{solve for } s_i \in \mathbb{R})$$

③ $$\underline{u}_t = \underset{\substack{\uparrow \\ \text{not transpose!!} \\ \text{Just } \underbrace{A \cdot A \cdots A}_{t}}}{A^t} \underline{u}_0 = A^t \left(\sum_{i=1}^{n} s_i \cdot \underline{v}_i \right) = \sum_{i=1}^{n} s_i (A^t \underline{v}_i)$$

$$= \sum_{i=1}^{n} s_i \lambda_i^t \underline{v}_i$$

④ $\lim_{t \to \infty} \underline{u}_t$: Terms where $|\lambda_i| < 1 \to 0$

Terms where $|\lambda_i| > 1 \to \infty$

Terms where $\lambda_i = 1$ gives $s_i 1^t \underline{v}_i$.

Equilibrium $\Leftrightarrow |\lambda_i| \le 1 \ \forall i$, then $\lim_{t \to \infty} \underline{u}_t = \sum_{i \in I} s_i \underline{v}_i$

for $I = \{ i \mid \lambda_i = 1 \}$