Problem Sheet 3
DRE 7007 Mathematics

BI Norwegian Business School

Solutions Problem Sheet 3

1. Since $f(x) = x\sin(1/x)$ for $x \neq 0$, it is continuous and differentiable in $U = \{x : x \neq 0\}$ with

$$f'(x) = 1\cdot\sin(1/x) + x\cdot\cos(1/x)\cdot(-1/x^2) = \sin(1/x) - \frac{1}{x}\cos(1/x)$$

Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\cdot\sin(1/x) = \underline{0} \qquad (x \to 0,\ -1 \leq \sin(1/x) \leq 1)$$

f is <u>continuous</u> at $x=0$. We have

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(1/h) - 0}{h} = \lim_{h\to 0}\sin(1/h)$$

which does not exist. Hence f is <u>not differentiable</u> at $x=0$.
It is then <u>not C^1</u> on $\mathbb{R}$.

A similar argument for $g(x) = \begin{cases} x^2\sin 1/x, & x \neq 0 \\ 0, & x = 0\end{cases}$ shows that

g is <u>continuous</u> since $\lim_{x\to 0} x^2\sin 1/x = 0$, that

$$g'(0) = \lim \frac{h^2\sin 1/h - 0}{h} = \lim_{h\to 0} h\sin 1/h = 0$$

So that g is <u>differentiable</u>, with $g'(x) = 2x\sin 1/x + x^2\cdot\cos(1/x)\cdot(-1/x^2)$
$= 2x\sin 1/x - \cos 1/x$

Since $\lim_{x\to 0} g'(x) = \lim_{x\to 0} 2x\sin 1/x - \cos 1/x = -\lim_{x\to 0}\cos 1/x$ does not exist,

g is <u>not C^1</u> on $\mathbb{R}$.

2. a)

$$D^2f(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & -\frac{1}{4}y^{-3/2}\end{pmatrix} \Rightarrow D^2f(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & -1/4\end{pmatrix}$$

b)

$$D^2f(x,y,z) = \begin{pmatrix} -\frac{1}{4}x^{-3/2} & 0 & 0 \\ 0 & -\frac{1}{4}y^{-3/2} & 0 \\ 0 & 0 & -\frac{1}{4}z^{-3/2}\end{pmatrix} \Rightarrow D^2f(2,2,2) = \begin{pmatrix} -\frac{1}{8\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{8\sqrt{2}} & 0 \\ 0 & 0 & -1/8\sqrt{2}\end{pmatrix}$$

c) $$D^2 f(x,y,z) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = D^2 f(1,1,1)$$

3. a) <u>Indefinite</u> since $D_1 = 2$, $D_2 = -1/2$.

b) <u>Negative definite</u> since $D_1 = -\frac{1}{8\sqrt{2}}$, $D_2 = \frac{1}{128}$, $D_3 = -\frac{1}{1024\sqrt{2}}$

c) <u>Indefinite</u> since $D_2 = -1$.

4. a) $$D^2 f(x,y) = \begin{pmatrix} y^2 & 1+xy \\ 1+xy & x^2 \end{pmatrix} \cdot e^{xy}$$

b) $$D^2 f(x,y,z) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

5. a) $$\left.\begin{array}{l} D_1 = y^2 e^{xy} \geq 0 \\ \Delta_1 = x^2 e^{xy} \geq 0 \\ D_2 = (-1-2xy)\, e^{2xy} \end{array}\right\} \begin{array}{l} -1-2xy \geq 0: \text{ pos. semidefinite} \\ -1-2xy < 0: \text{ indefinite} \\ (-1-2xy > 0: \text{ pos. defin.}) \end{array}$$

b) $$\left.\begin{array}{lll} D_1 = 0 & D_2 = -z^2 & D_3 = 2xyz \\ \Delta_1 = 0 & \Delta_2 = -y^2 & \\ \Delta_1 = 0 & \Delta_2 = -x^2 & \end{array}\right\} \begin{array}{l} (x,y,z) \neq (0,0,0): \text{ indefinite} \\ (x,y,z) = (0,0,0): \text{ pos. semidefinite and} \\ \qquad\qquad\qquad\quad \text{ neg. — '' —} \end{array}$$

6. Let $f(x,y) = \begin{cases} 0 , & (x,y)=(0,0) \\ \dfrac{xy(x^2-y^2)}{x^2+y^2} , & (x,y)\neq(0,0) \end{cases}$. Define $U \subseteq \mathbb{R}^2$ to be the open set $U = \{(x,y) : (x,y) \neq (0,0)\}$. Then f is clearly C^2 on U, being a rational function defined on U.

We first prove that f is a C^1 function on $\mathbb{R}^2$. This implies that it is continuous on $\mathbb{R}^2$.

For $(x,y) \in U$,

$$\frac{\partial f}{\partial x} = \frac{(3x^2-y^3)(x^2+y^2) - (x^3y - xy^3)\cdot 2x}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \quad \text{(by symmetry)}$$

For $(x,y) = (0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$$

It remains to prove that $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial y} = 0$.

$\lim\limits_{\underline{x}\to 0} \frac{\partial f}{\partial x} = 0$: For any $\varepsilon > 0$, there is $\delta > 0$ s.t. $\|\underline{x}\| < \delta \Rightarrow \left|\frac{\partial f}{\partial x} - 0\right| < \varepsilon$

$$\left|\frac{\partial f}{\partial x}\right| = \left|\frac{x^4y + 4x^2y^3 - y^5}{x^2+y^2}\right| = |y| \cdot \left|\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}\right| \leq |y| \cdot \left|\frac{x^4+4x^2y^2+4y^4}{(x^2+y^2)^2}\right|$$

$$= |y| \cdot \frac{(x^2+2y^2)^2}{(x^2+y^2)^2} \leq |y| \cdot 2^2 = 4|y| \leq 4 \cdot \|(x,y)\|$$

If $\delta \leq \varepsilon/4$, then $\|(x,y)\| < \delta \Rightarrow \left|\frac{\partial f}{\partial x}\right| \leq 4\delta < \varepsilon$. ok.

$\lim\limits_{\underline{x}\to 0} \frac{\partial f}{\partial y} = 0$: Similarly, this follows since we have

$$\left|\frac{\partial f}{\partial y}\right| = |x| \cdot \left|\frac{x^4 - 4x^2y^2 - y^4}{x^2+y^2}\right| = |x| \cdot \left|\frac{y^4+4y^2x^2-x^4}{(y^2+x^2)^2}\right| \leq |x| \cdot \frac{(y^2+2x^2)^2}{(y^2+x^2)^2} \leq 4|x|.$$

We compute the second order partial derivatives at $(x,y)=(0,0)$:

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{\partial f/\partial x(h,0) - \partial f/\partial x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \underline{0}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{h\to 0} \frac{f'_x(0,h) - f'_x(0,0)}{h} = \lim_{h\to 0} \frac{-h}{h} = \underline{-1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{h\to 0} \frac{f'_y(h,0) - f'_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = \underline{1}$$

$$\frac{\partial^2 f}{\partial y^2} = \lim_{h\to 0} \frac{f'_y(0,h) - f'_y(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \underline{0}$$

(All partial derivatives at $(0,0)$; $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$)

Since $\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0)$, f is $\underline{\text{not } C^2 \text{ on } \mathbb{R}^2}$. Of course, f is C^2 on $U = \{(x,y) : (x,y) \neq (0,0)\}$ and the Hessian is given by straight-forward derivation:

$$H^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & (x,y) = (0,0) \\[2ex] \begin{pmatrix} \dfrac{12xy^5 - 4x^3y^3}{(x^2+y^2)^3} & \dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \\[2ex] \dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & \dfrac{-12x^5y+4x^3y^3}{(x^2+y^2)^3} \end{pmatrix}, & (x,y) \neq (0,0) \end{cases}$$