Plan:

Optimal control theory in continuous time

Functional:

Ex: $I = [0,1]$

$$C(I,\mathbb{R}) \xrightarrow{J} \mathbb{R}$$
$$f \longmapsto J(f) = \int_0^1 f(t)\,dt$$

$$C^2(I,\mathbb{R}) \xrightarrow{J} \mathbb{R}$$
$$y(t) = y \longmapsto J(y) = \int_0^1 \ln(y^2 + y'^2 + 1)\,dt$$

Optimal control theory: (cont. time)

max/min for functionals (with constraints)

Problem:

max $\int_{t_0}^{t_1} f(t,x,u)\,dt$ when $\begin{cases} x(t_0) = x_0 \\ x' = g(t,x,u) \\ u \in \mathcal{U} \subseteq \mathbb{R} \end{cases}$

either $x(t_1) = x_1$ a) / $x(t_1)$ free b)

$u = u(t)$ : control variable $\quad \mathcal{U} \subseteq \mathbb{R}$ control region

$x = x(t)$ : state variable

$u$ chosen $\Rightarrow$ can determine $x = x(t)$ using diff. eqn. and initial cond.

Special case: calculus of variations $\boxed{u = x'}$

$$\max \int_{t_0}^{t_1} F(t,x,x')\,dt \quad \text{when} \begin{cases} x(t_0) = x_0 \\ x(t_1) = x_1 \end{cases}$$

$\Downarrow$

Method: Euler eqn: $x^*$ optimal $\Rightarrow$ Euler eqn. satisfied: $F'_x = \frac{d}{dt}(F'_{x'})$

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We return to the optimal control problem:

$$\max \int_{t_0}^{t_1} f(t,x,u)\,dt \quad \text{when} \begin{cases} x(t_0) = x_0 \\ x' = g(t,x,u) \\ u \in \mathcal{U} \\ \text{a) } x(t_1) = x_1 \\ \text{b) } x(t_1) \text{ free} \end{cases}$$

Method: Hamiltonian

$$H = p_0 \cdot f(t,x,u) + p \cdot g(t,x,u)$$

where $p = p(t)$, $p_0 = 0$ or $p_0 = 1$

Pontryagin's maximum principle (necessary cond.)

If $(x^*, u^*)$ is optimal, then there is $p(t)$, $p_0$ with $(p(t), p_0) \neq (0,0)$ for all $t \in [t_0, t_1]$ such that

a) $u \mapsto H(t, x^*, u, p)$ has $u^*$ as maximizer

b) $p' = -H'_x(t, x^*, u^*, p)$

c) $\begin{cases} \text{a) } x(t_1) = x_1: & \text{no condition} \\ \text{b) } x(t_1) \text{ free}: & p(t_1) = 0 \end{cases}$

Thm: (Mangasarian)

Assume $(x^*, u^*)$ satisfies maximum principal with $p_0 = 1$. If $U$ is convex and

$$(x, u) \mapsto H(t, x, u, p)$$

is concave for all $t \in [t_0, t_1]$, then $(x^*, u^*)$ is optimal.

**Ex I:** $\max \int_0^T (1 - tx - u^2)\,dt$, $\quad \dot{x} = u$, $x(0) = x_0$, $U = \mathbb{R}$ $\quad$ ($x_0, T$ given)

**Necessary condition:**

$H = p_0 \cdot (1 - tx - u^2) + p \cdot u$

**$p_0 = 0$:** $p(t) \neq 0$

(B) $\dot{p} = 0 \Rightarrow p(t) = c \neq 0$ constant

$H = pu = c \cdot u$ has no max as a fn. in $u$, so (A) not satisfied

$\Downarrow$

no solutions with $p_0 = 0$

**$p_0 = 1$:** $H = 1 - tx - u^2 + pu$

$\dot{x} = u$

$x(0) = x_0$

(A) $\dfrac{\partial H}{\partial u} = p - 2u = 0 \Rightarrow u = \frac{1}{2}p$

(B) $p' = -(-t) = t \Rightarrow p = \frac{1}{2}t^2 + C$

(C) $p(T) = 0 \Rightarrow C = -\frac{1}{2}T^2 \Rightarrow p(t) = \underline{\frac{1}{2}t^2 - \frac{T^2}{2}} \Rightarrow u = \underline{\frac{1}{4}t^2 - \frac{T^2}{4}}$

$\dot{x} = u = \frac{1}{4}t^2 - \frac{T^2}{4} = \frac{1}{12}t^3 - \frac{T^2}{4}t + C$

$x(0) = x_0 \Rightarrow C = x_0 \Rightarrow x = \underline{\frac{1}{12}t^3 - \frac{T^2}{4}t + x_0}$

$U = \mathbb{R}$ convex & $H = 1 - tx - u^2 + pu$ concave in $(x, u)$ $\Rightarrow$ $u = \frac{1}{4}t^2 - T^2/4$, $x = \frac{1}{12}t^3 - \frac{T^2}{4}t + x$ is the maximizer