Plan:

1. Optimization: Generalities / Intro
2. Unconstrained optimization

Optimization problem:

$$\max_{\underline{x} \in D} f(\underline{x}) = f(x_1, \ldots, x_n), \quad \text{where } D \subseteq \mathbb{R}^n$$

$$\min_{\underline{x} \in D} f(\underline{x}) = f(x_1, \ldots, x_n), \quad \text{— '' —}$$

Setup: $D \subseteq \mathbb{R}^n$

$f : D \to \mathbb{R}$ continuous

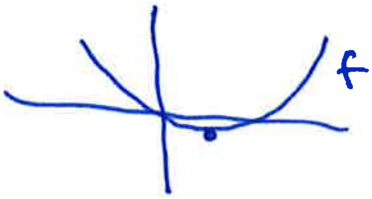
A solution to the max problem is a pt. $\underline{x}^* \in D$ such that $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x} \in D$.

$\arg\max \{ f(\underline{x}) : \underline{x} \in D \}$ is the set of solutions to the max propblem.

this set might be empty, have one element or several elements.

Similar defn. for min.

$$\arg\min\{f(\underline{x}) : \underline{x}\in D\} = \arg\max\{-f(\underline{x}) : \underline{x}\in D\}$$



The attainable values of $\max_{\underline{x}\in D} f(\underline{x})$ is the set

$$f(D) = \{f(\underline{x}) : \underline{x}\in D\} \subseteq \mathbb{R}.$$

$$\boxed{\max_{\underline{x}\in D} f(\underline{x}) \text{ has solutions } \iff \sup f(D) \in f(D)}$$

D : constraint set

f : objective fn.

Result: If $\varphi : \mathbb{R}\to\mathbb{R}$ is ~~[illegible]~~ strictly increasing fn. (that is, $x<y \Rightarrow f(x)<f(y)$), then

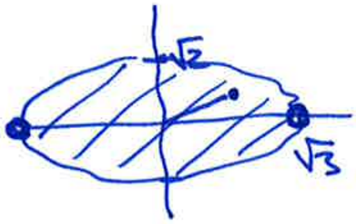
$\max_{\underline{x}\in D} f(\underline{x})$ has the same solutions as $\max_{\underline{x}\in D} \varphi(f(\underline{x}))$

$$\arg\max\{f(\underline{x}) : \underline{x}\in D\} = \arg\max\{\varphi(f(\underline{x})) : \underline{x}\in D\}$$

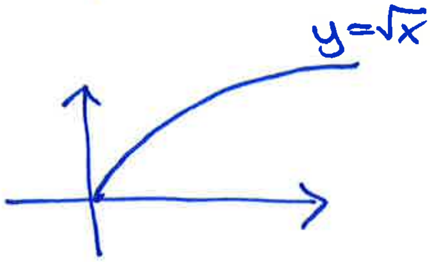
max $f(x,y)=\sqrt{x^2+y^2}$ when $2x^2+3y^2 \le 6$

$D = \{(x,y) : 2x^2+3y^2 \le 6\}$

$\updownarrow$ same solutions



max x^2+y^2 when $2x^2+3y^2 \le 6$



Results:

① Extreme value thm (Weierstrass' thm):

f cont., D compact $\Rightarrow$ f(D) compact (max $f(\underline{x})$, $\underline{x} \in D$ has a sol'n.)

f not cont. or D not compact $\Rightarrow$ there may or may not be a sol'n.

② Convex/concave opt:

② Unconstrained optimization

Defn: A maximizer $\underline{x}^*$ of $\max_{\underline{x} \in D} f(\underline{x})$ is called

- unconstrained if $\underline{x}^* \in D$ is an interior pt. of D (that is, $\underline{x}^* \notin \partial D$).
- constrained if $\underline{x}^* \in \partial D$ is a boundary pt.

Cases:

i) $D = \mathbb{R}^n$ or $D \subseteq \mathbb{R}$ is open: there can only be unconstrained max (min)

ii) $D \subseteq \mathbb{R}^n$ not open:
- constrained max/min
- unconstrained —"—

A pt. $\underline{x} \in D$ is called a local max. for f if there is an open ball $B(\underline{x}, \varepsilon)$ such that $f(\underline{x}') \leq f(\underline{x})$ for all $\underline{x}' \in B(\underline{x}, \varepsilon)$.

Similar for local min.

$$\boxed{\underline{x}^* \text{ maximum for } f \text{ (min.)} \Rightarrow \underline{x}^* \text{ local maximum (min.)}}$$

Methods: Unconstrained optimization

$$\max_{\underline{x} \in D} f(\underline{x})$$

(f continuous)

$f : D \to \mathbb{R}$

Defn: A stationary pt for f is a pt. $\underline{x} \in D$ such that

$$Df(\underline{x}) = 0$$

$$\Updownarrow$$

f is differentiable and

first order conditions (FOC) $\longrightarrow$ $\boxed{\frac{\partial f}{\partial x_1}(\underline{x}) = \dots = \frac{\partial f}{\partial x_n}(\underline{x}) = 0}$

A critical pt for f is a pt. $\underline{x} \in D$ such that

$Df(\underline{x}) = 0$ or $Df(\underline{x})$ does not exist

If f is C^1, then critical pts = stationary pts.

Result: If $\underline{x}^*$ is a local max for the optimization problem

$$\max_{\underline{x} \in D} f(\underline{x})$$

then one of the following holds:

i) $\underline{x}^*$ is a critical interior pt. of D

or

ii) $\underline{x}^*$ is a boundary point of D.

$\max/\min_{\underline{x} \in D} f(\underline{x})$:

Consider FOC: $\boxed{\begin{matrix} f'_{x_1} = 0 \\ \vdots \\ f'_{x_n} = 0 \end{matrix}}$ $\rightarrow$ candidates for max/min

consider critical pts that are not stationary $\rightarrow$ —''—

When we make the list of candidate pts, check that all point are in D (and not in ∂D for unconstrained max/min).

Second order conditions:

can determine (in most cases) the local nature of the stationary pt.

Second derivative test:

If $\underline{x}^*$ is a stationary pt for f, then

$H(f)(\underline{x}^*)$ pos. defn. $\Rightarrow$ $\underline{x}^*$ local min

$H(f)(\underline{x}^*)$ neg. defn. $\Rightarrow$ $\underline{x}^*$ local max

$H(f)(\underline{x}^*)$ indefn. $\Rightarrow$ $\underline{x}^*$ saddle pt.

(i.e. stationary pt. that is neither local max/min)

Convex/concave optimization

f concave $\Rightarrow$ $\begin{cases} \text{argmax}\{f(x) : x \in D\} \text{ is empty or a } \underline{\text{convex set}} \\ \text{any stationary pt. of } f \text{ is a maximizer} \end{cases}$

f strictly concave $\Rightarrow$ $\begin{cases} \text{argmax}\{f(x) : x \in D\} \text{ is empty or } \underline{\text{a pt.}} \\ \text{any stationary pt. of } f \text{ is a max.} \end{cases}$

f strictly quasi-concave $\Rightarrow$ $\begin{cases} \text{argmax}\{f(x) : x \in D\} \text{ is empty or } \underline{\text{a pt.}} \\ \text{any local max is a global max.} \end{cases}$

Similar results hold for convex/quasi-convex fn. and min.
A good reference is [S] Ch. 7-8.

__Defn:__ f strictly quasi concave if: $f(\lambda \cdot \underline{x} + (1-\lambda)\underline{x}') > \min\{f(\underline{x}), f(\underline{x}')\}$ for all $\lambda \in (0,1)$ and all $\underline{x}, \underline{x}'$.

(f quasi-concave $\Leftrightarrow$ $f(\lambda \underline{x} + (1-\lambda)\underline{x}') \geq \min\{f(\underline{x}), f(\underline{x}')\}$ for all $\lambda \in [0,1]$ and all $\underline{x}, \underline{x}'$)

Alternative formulation of quasi-concavity