Plan:

1. Euclidean space and general vector spaces (inner products, norms, metrics)
2. Sequences
3. Topology

$\mathbb{R}$ = real numbers

1 Vector spaces

A vector space is a set V of elements called vectors, together with

a) addition: $V \times V \xrightarrow{+} V$
$(\underline{v}, \underline{w}) \mapsto \underline{v} + \underline{w}$

b) scalar multiplication: $\mathbb{R} \times V \to V$
$(r, \underline{v}) \mapsto r \cdot \underline{v}$

such that

i) $(u+v)+w = u+(v+w)$

ii) there is a zero vector $0 \in V$ such that $u + 0 = u$

iii) for each vector $v \in V$, there is a vector $-\underline{v}$ such that $v + (-v) = 0$

iv) $u+v = v+u$

v) $r \cdot (sv) = (rs) \cdot v$

vi) $(r+s)v = rv + sv$

vii) $r \cdot (v+w) = rv + rw$

viii) $1 \cdot v = v$

Euclidean space:

$V = \mathbb{R}^n = \{\underline{v} : n\text{-vectors}\}$

$= \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} : v_1, v_2, \ldots, v_n \in \mathbb{R} \right\}$

is a vector space with

$$\underline{v} + \underline{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

$$r \cdot \underline{v} = r \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

Ex: Let $V = C(I, \mathbb{R})$ where $I = [0,1]$
$= \{f : I \to \mathbb{R} \text{ cont. fn.}\}$

$e^x : [0,1] \to \mathbb{R}$
x^2
$\sqrt{x}$
$\vdots$

$V = C(I, \mathbb{R})$ is a vector space with usual addition / scalar multiplication

Inner product spaces

An inner product on a vector space V is a product

$$V \times V \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \langle v, w \rangle = v \cdot w$$

such that

i) $\langle v, w \rangle = \langle w, v \rangle$

ii) $\langle a\underline{u} + b\underline{v}, \underline{w} \rangle = a \langle \underline{u}, \underline{w} \rangle + b \cdot \langle \underline{v}, \underline{w} \rangle$

iii) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow \underline{v} = \underline{0}$

Euclidean space:

$V = \mathbb{R}^n$ has the std. Euclidean inner product:

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Cauchy-Schwarz inequality:

If V is an inner product space = vectorspace with a given inner product, then

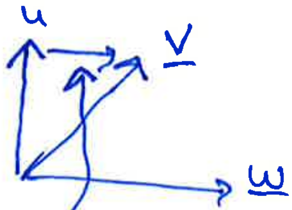
$$|\langle v, w\rangle| \leq \langle v, v\rangle^{1/2} \cdot \langle w, w\rangle^{1/2}$$

for all $\underline{v}, \underline{w} \in V$, and equality holds if and only if $\underline{v}, \underline{w}$ are linearly dependent.

Proof: Assume $\underline{w} \neq \underline{0}$:

$$\boxed{u = v - \frac{\langle v, w\rangle}{\langle w, w\rangle} \cdot w}$$

$$\langle u, w\rangle = \langle v, w\rangle - \frac{\langle v, w\rangle}{\cancel{\langle w, w\rangle}} \cancel{\langle w, w\rangle} = 0$$



$$\underline{v} = \underline{u} + \underbrace{\frac{\langle v, w\rangle}{\langle w, w\rangle}} \cdot \underline{w}$$

$$\langle v, v\rangle = \langle u + rw, u + rw\rangle$$
$$= \langle u, u\rangle + r\underline{\langle w, u\rangle} + \underline{\langle u, rw\rangle} + \langle rw, rw\rangle$$
$$= \langle u, u\rangle + r^2 \cdot \langle w, w\rangle \geq r^2 \cdot \langle w, w\rangle$$
$$= \frac{\langle v, w\rangle^2}{\langle w, w\rangle^{\cancel{2}}} \cdot \cancel{\langle w, w\rangle} = \frac{\langle v, w\rangle^2}{\langle w, w\rangle}$$

$$\langle v,v\rangle \geq \frac{\langle v,w\rangle^2}{\langle w,w\rangle}$$

$$\langle v,v\rangle \cdot \langle w,w\rangle \geq \langle v,w\rangle^2$$

$$|\langle v,w\rangle| \leq \langle v,v\rangle^{1/2} \cdot \langle w,w\rangle^{1/2} \leftarrow \text{CS}$$

CS implies $\dfrac{|\langle v,w\rangle|}{\|v\|\cdot\|w\|} \leq 1$

Equality holds $\Leftrightarrow$ $\langle u,u\rangle = 0$

$u = 0$

$$v - \frac{\langle v,w\rangle}{\langle w,w\rangle}\cdot w = 0$$

$$v = \frac{\langle v,w\rangle}{\langle w,w\rangle}\cdot w$$

lin. dependent. $\square$

Definitions for inner product spaces

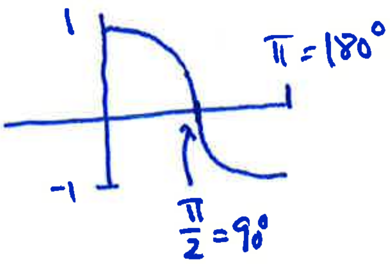
$\|v\| = \sqrt{\langle v,v\rangle}$ ← Axioms: $\|v\| \geq 0$ and $\|v\| = 0$ only if $v = 0$

length of v

$\alpha = \angle(v,w)$ by $\cos\alpha = \dfrac{\langle v,w\rangle}{\|v\|\cdot\|w\|}$

↑ number between -1 and 1

angle between v, $w \neq 0$, a unique number between $0°$ and $180°$.



Note: $v \perp w$ if $\langle v,w\rangle = 0$ ($\alpha = 90°$)

$\alpha = 0°$: v has same dir as w

$\alpha = 180°$: v has opposite dir of w

Norm: V vector space

$\|w\|$ length fn.

s.t.

Norm: $V \to \mathbb{R}$

$\underline{w} \mapsto \|\underline{w}\|$

i) $\|x\| \geq 0$ for all $\underline{x}$, and $\|\underline{x}\| = 0$ only if $\underline{x} = 0$

ii) $\|r\underline{x}\| = |r| \cdot \|x\|$

iii) $\|\underline{x} + y\| \leq \|x\| + \|y\|$

inner product $\Rightarrow$ norm

$\langle -, - \rangle$ $\qquad \|v\| = \sqrt{\langle v, v \rangle}$

Sup norm: $V = C(I, \mathbb{R})$

$$\|f\|_{sup} = \sup\{f(x) : x \in I\} = \sup_{x \in I} f(x)$$

$J \subseteq \mathbb{R}$: an upper bound for J is M s.t.

$M \geq j$ for any $j \in J$

"supremum" $\to$ $\sup J :=$ least upper bound for J.

"infimum" $\to$ $\inf J :=$ greatest lower bound for J

- $J = \{1/n : n = 1, 2, 3, \ldots\} = \{1/1, 1/2, 1/3, \ldots\}$ $\qquad \sup J = 1$, $\inf J = 0$

X can be any space (set), does not have to be a vector space.

Metric space: ~~Vector space~~ X with distance function

(X, d) $\quad d(\underline{x}, \underline{y})$ satisfying axioms: i) - iii)

See [ES] 2.4

inner product $\Rightarrow$ norm $\Rightarrow$ distance fn. = metric

$\langle -, - \rangle \dashrightarrow \|v\| = \sqrt{\langle v, v \rangle}$

$\| - \| \dashrightarrow d(u,v) = \|u - v\|$

② **Sequences:** (X, d) metric space

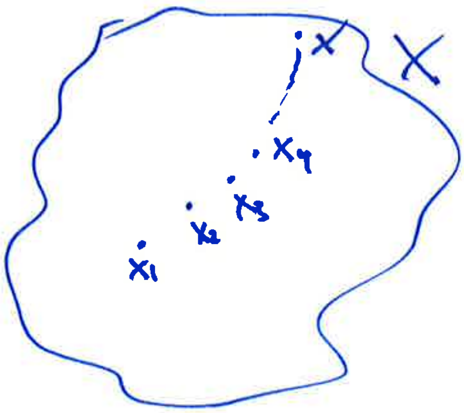
Typical example

$X = \mathbb{R}^n$

$d(\underline{x}, \underline{y}) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

Sequence:

$\{x_i : i = 1, 2, \dots\} \subseteq X$



Defn:

$\{x_i\} \to x$ (the seq. converges to $x \in X$)

if

"$\lim_{i \to \infty} x_i = x$"

or more precisely:

for each $\varepsilon > 0$, there exists an N s.t.

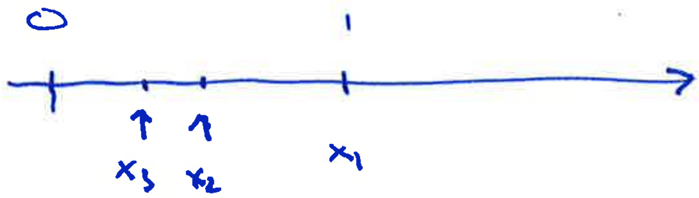
$i \geq N \Rightarrow d(x_i, x) < \varepsilon$

Ex: $V = \mathbb{R}$ with std metric $d(x,y) = \sqrt{(x-y)^2} = |x-y|$

$x_i = 1/i \, , \, i \geq 1$

$x_1 = 1 \quad x_2 = 1/2 \quad x_3 = 1/3 \quad \ldots$

$\boxed{\{x_i\} \text{ conv. to } x = 0}$



Since if $\varepsilon > 0$ is given, we can choose $N = 1/\varepsilon$

$i \geq N = \frac{1}{\varepsilon}$

$$d(x_i, 0) = d(x_i, x) = |x_i| = \frac{1}{i} \leq \frac{1}{N} = \varepsilon$$

Cauchy sequence:
If (X,d) is a metric space, then a sequence (x_i) in X is called a Cauchy sequence if the following holds:

For each $\varepsilon > 0$, there exist N such that

$$i,j \geq N \Rightarrow d(x_i, x_j) < \varepsilon$$

Results:

1. If (x_i) converges to x, then (x_i) is Cauchy.
2. If (x_i) is a Cauchy seq. in Euclidean space, then $(x_i) \to x$ for some $\underline{x} \in \mathbb{R}^n$.

(X, d) is complete if any Cauchy seq. in X converges to some $x \in X$.

2 : $(\mathbb{R}^n, d)$ is complete.

3. If (x_i) converges, then it is bounded, i.e. there is an open ball $B(\underline{p}, R) = \{\underline{x} \in X : d(\underline{x}, \underline{p}) < R\}$ that contains $\{x_i : i\}$.