Plan:

① Example: Infinite time horizon (see end of notes)

② Fixed points and fixed point theorems

③ Correspondences

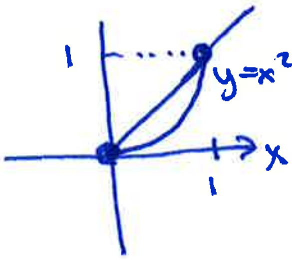
② Fixed points:

$f: X \to X$ operator (fn.)

$x \in X$ fixed point: $f(x) = x$

Ex: $f(x) = x^2$ as $f: [0,1] \to [0,1]$

$x \mapsto x^2$

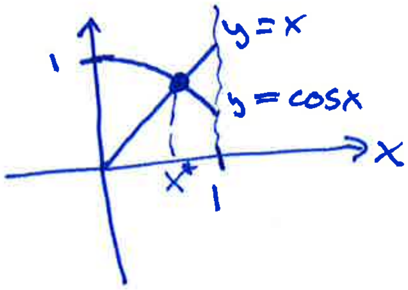


$f(x) = x^2 = x$

$\underline{x=0}, \underline{x=1}$

Ex: $f: [0,1] \to [0,1]$

$x \mapsto \cos x$



one fixed pt. x^*

cannot find x^* analytically

$\boxed{\text{Any cont fn. } f: [0,1] \to [0,1] \text{ has a fixed pt.}}$

Ex: A : $n\times n$-matrix

$$A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\underline{x} \longmapsto A\cdot\underline{x}$$

Fixed point of A: $A\underline{x} = \underline{x}$

$\shortparallel$

Eigenvectors for $\lambda = 1$.

Fixed points = equilibrium states

Ex: Markov chain $\underline{x}_{t+1} = A\cdot\underline{x}_t$

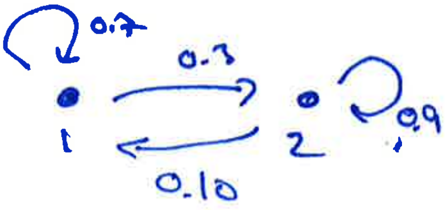
A: transition matrix

Equilibrium: $\underline{x} = A\cdot\underline{x}$

$\shortparallel$

fixed point.

$$A = \begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}$$



Fixed pts: $A\underline{x} = \underline{x}$

$(A-I)\underline{x} = \underline{0}$

$$\left(\begin{array}{cc|c} -0.3 & 0.1 & 0 \\ 0.3 & -0.1 & 0 \end{array}\right)$$

$-3x + y = 0$

y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/3 \\ y \end{pmatrix} = \frac{y}{3}\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

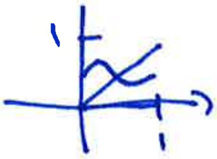
Equilibrium: $\text{span}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Interpretation: 25% state 1
75% state 2

Brouwer's fixed point thm:

If $f: K \to K$ is a continuous map, and K is a non-~~empty~~empty, compact and convex set, then f has a fixed point.

Ex: $K = [0,1]$

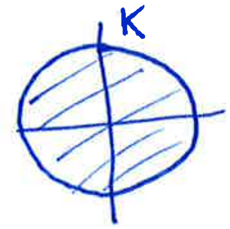


– compact in $\mathbb{R}^n$ = closed and bounded

Ex: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+1$ } no fixed pt. not bounded

$f: (0,1) \to (0,1)$, $x \mapsto \frac{x+1}{2}$ } no fixed pt. not closed $(x=1)$

Ex:

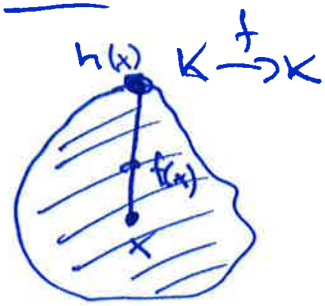


f: rotation by 45°

no fixed points

not convex, has holes

Proof:



$K \xrightarrow{f} K$

$$\begin{tikzcd} \pi_1: \mathbb{Z} \arrow[r] \arrow[rr, bend left, "id"] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

not possible

Assume no fixed pts.

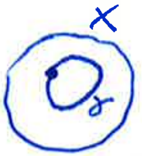
$h: K \to K$

$h(x) = x$ if $x \in \partial K$.

$$\begin{tikzcd} \partial K \arrow[r, "i"] \arrow[rr, bend right, "id"'] & K \arrow[r, "h"] & \partial K \end{tikzcd}$$

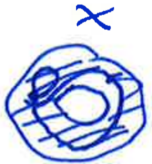
Fundamental group: $\pi_1(X)$

$$\pi_1(X) \; ``=" \; \mathbb{Z}^{\#\text{holes in } X}$$



no holes

$\pi_1(X) = 0$



$\pi_1(X) = \mathbb{Z}$

Brouwer's fixed point thm

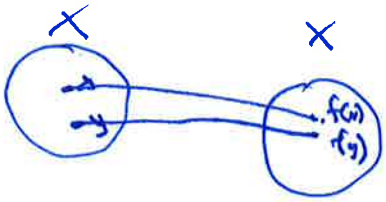
K: non-empty, compact, convex

$f: K \to K$ cont. } $f: K \to K$ has a fixed point

Note: - non-constructive, no information about how to find the fixed pt.

- may be more than one fixed pt

Fixed point thm for contractions



(X, d) : metric space

$f: X \to X$ is called a contraction if there is a β with $0 < \beta < 1$

s.t. $\boxed{d(f(x), f(y)) \leq \beta \cdot d(x, y)}$ $\Longrightarrow$ $d(f(x), f(y)) < d(x,y)$

for all $x, y \in X$.

Ex: $f(x) = \frac{1}{2}x$ $\quad f: \mathbb{R} \to \mathbb{R}$

$$d(f(x), f(y)) = d(\tfrac{1}{2}x, \tfrac{1}{2}y) = |\tfrac{1}{2}x - \tfrac{1}{2}y|$$

$$= \tfrac{1}{2} \cdot |x - y| \quad \text{contraction } \beta = 1/2$$

All contractions are continuous.

Fixed point thm for contractions

If X is complete metric space and $f : X \to X$ is a
$\overset{\displaystyle X}{\underset{(X,d)}{\shortparallel}}$ non-empty

contraction, then f has a uniqu fixed point.

Recall: complete = all Cauchy-sequences converge

Construction:

$$\left.\begin{array}{l} x_0 \in X \\ x_1 = f(x_0) \\ x_2 = f(x_1) \\ \vdots \end{array}\right\}$$

$x_0, x_1, x_2, \ldots$ sequence in X

Cauchy-seq.

$\Downarrow$ complete

$x = \lim_i x_i \in X$

$f(x) = x$ and this is the unique fixed pt.

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \leq \beta \cdot d(x_0, x_1)$$

$$d(x_2, x_3) \leq \beta \cdot d(x_1, x_2) \leq \beta^2 d(x_0, x_1)$$

$$\vdots$$

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k})$$

$$\leq \beta^n \cdot d(x_0 x_1) + \beta^{n+1} d(x_0, x_1) + \ldots + \beta^{n+k-1} \cdot d(x_0, x_1) = \beta^n \cdot \frac{1-\beta^k}{1-\beta} d(x_0, x_1)$$

$$\leq \boxed{\frac{\beta^n}{1-\beta}} d(x_0, x_1)$$

$\to 0$ as $n \to \infty$

$$f(x) = f(\lim x_i) = \lim_i f(x_i) = x$$

Assume x, y are fixed pts: $d(x,y) = d(f(x), f(y)) \leq \beta \cdot d(x,y)$

$\Rightarrow d(x,y) = 0 \Rightarrow x = y$.

Ex: $S \subseteq \mathbb{R}^n$ compact

$X = C(S, \mathbb{R})$ all cont. maps from S to $\mathbb{R}$

metric space with sup-norm and corr. metric $\quad d(f,g) = \sup_{x \in S} |f(x) - g(x)|$

$X = C(S,\mathbb{R})$ with sup is complete

Ex: $S \subseteq \mathbb{R}^n$

$B(S,\mathbb{R}) = \{ f : S \to \mathbb{R}$ cont. and bounded$\}$

f bounded: there is $M > 0$ s.t. $|f(x)| < M$ for all $x \in S$

$B(S,\mathbb{R})$ is complete (sup-norm)

Application: Bellman equation $\quad J(x) = \max \{ f(x,u) + \beta \cdot J(\underset{g(x,u)}{x}) \}$

Assume: $0 < \beta < 1$, f bounded, f, g cont.

$$X = B(\mathbb{R},\mathbb{R}) \xrightarrow{T} X = B(\mathbb{R},\mathbb{R})$$

$$J \longmapsto x \mapsto \max_{u \in U} \{ f(x,u) + \beta \cdot J(g(x,u)) \}$$

T well-defined contraction

J fixed pt of T = sol. of Bellman eqn. J

$J(T) = J$

Conclusion: Assuming, there is a unique bounded Solution of the Bellman eqn.

③ Correspondences

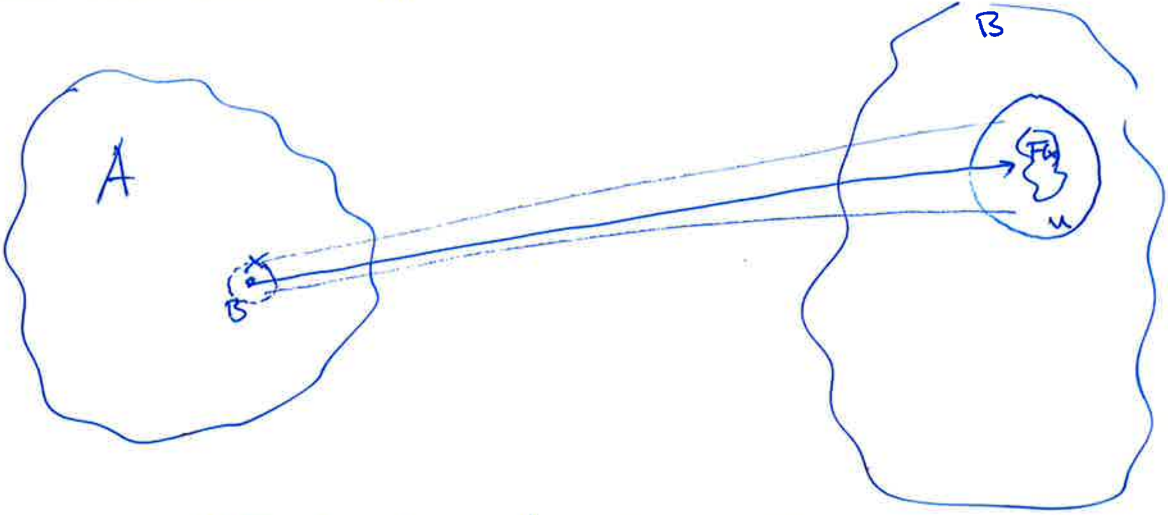
A **correspondence** $F: A \twoheadrightarrow B$ is a specification of a subset $F(x) \subseteq B$ for all $x \in A$. That is, a multivalued function.

The graph of F is

$$\Gamma(F) = \{(x,y) \in A \times B : x \in A, y \in F(x)\}$$

F is **upper hemicontinuous** if the following condition is satisfied:

For any $x \in A$, and for any open set U containing $F(x)$, there is an open ball $B(x,r)$ around x such that $F(x') \subseteq U$ for all $x' \in B(x,r) \cap A$.



If the graph of F is compact, then F is upper hemicont.

Theorem (Kakutani)

If $K \subseteq \mathbb{R}^n$ is nonempty, compact and convex and if $F: K \to K$ is upper hemicontinuous such that $F(x)$ is nonempty and convex for all $x \in K$, then F has a fixed point $x \in K$ (i.e. $F(x) \ni x$).

①

12.3.1. $\max \sum_{t=0}^{\infty} \beta^t \left(-e^{-u_t} - \frac{1}{2}e^{-x_t}\right)$ when $\begin{cases} x_0 \text{ given} \\ x_{t+1} = 2x_t - u_t \\ U = \mathbb{R} \end{cases}$

Put: $J(x) = -\alpha e^{-x}$ for $\alpha > 0$.

LHS: $\max_{u \in U} \{ f(x,u) + \beta J(2x-u) \} = \max_{u \in U} \{ -e^{-u} - \frac{1}{2}e^{-x} + \beta \cdot (-\alpha)\, e^{-(2x-u)} \}$

$$h(u) = -e^{-u} - \frac{1}{2}e^{-x} - \alpha\beta e^{-2x+u}$$

$$h'(u) = e^{-u} - \alpha\beta e^{-2x} \cdot e^{u} = \frac{1 - \alpha\beta e^{-2x} \cdot e^{2u}}{e^{u}} = 0$$

$$e^{2u} = \frac{1}{\alpha\beta e^{-2x}} = \frac{e^{2x}}{\alpha\beta} \Rightarrow e^{u} = \frac{e^{x}}{\sqrt{\alpha\beta}} = \frac{1}{\sqrt{\alpha\beta}} e^{x}$$

$$u = x + \ln\left(\frac{1}{\sqrt{\alpha\beta}}\right)$$

$$\underline{u = x - \frac{1}{2}\ln(\alpha\beta)}$$

$$h''(u) = -e^{-u} - \alpha\beta e^{-2x} e^{u} < 0 \Rightarrow u^* = x - \frac{1}{2}\ln(\alpha\beta)$$

$$\max_{u} h(u) = h(u^*) = -\frac{\sqrt{\alpha\beta}}{e^{x}} - \frac{1}{2}\frac{1}{e^{x}} - \alpha\beta e^{-x} e^{-\frac{1}{2}\ln(\alpha\beta)}$$

$$= e^{-x}\left(-\sqrt{\alpha\beta} - \frac{1}{2} - \alpha\beta \cdot \frac{1}{\sqrt{\alpha\beta}}\right)$$

$$= \left(-2\sqrt{\alpha\beta} - \frac{1}{2}\right)e^{-x} = -\alpha e^{-x} \leftarrow \text{RHS}$$

$$\alpha = 2\sqrt{\alpha\beta} + \frac{1}{2} : \quad (\sqrt{\alpha})^2 - 2\sqrt{\beta} \cdot (\sqrt{\alpha}) - \frac{1}{2} = 0$$

$$\sqrt{\alpha} = \frac{2\sqrt{\beta} \pm \sqrt{4\beta - 4 \cdot (-1/2)}}{2}$$

$$= \sqrt{\beta} \pm \sqrt{\beta + 1/2} = \underline{\underline{\sqrt{\beta} + \sqrt{\beta + 1/2}}}$$

$$\alpha = \beta + (\beta + 1/2) + 2\sqrt{\beta}\sqrt{\beta + 1/2}$$

$$= 2\beta + 1/2 + 2\sqrt{\beta^2 + 1/2\,\beta}$$

$$\alpha = \left(\sqrt{\beta} + \sqrt{\beta + 1/2}\right)^2$$

$$\underline{J = -\left(\sqrt{\beta} + \sqrt{\beta + 1/2}\right)^2 e^{-x}} \leftarrow$$

is solution