LECTURE 9

DRE 7007 MATHEMATICS

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PLAN:

1. Optimal control theory – discrete time, finite horizon
2. Infinite horizon

Reading:
[FMEA] 12
[S] 11–12

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① **Problem** (finite horizon)

$$\max_{(\min)} \sum_{t=0}^{T} f(t, x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(t, x_t, u_t) \text{ for } t = 0,1,\dots,T-1 \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

Comments:

- the set $U \subseteq \mathbb{R}$ of admissible controls may depend on the state variable x_t
- Sometimes $f(T, x_T, u_T) = S(x_T)$ and is called a scrap value function

A) **Dynamic programming and the Bellman equation**

Let $s \leq T$ and define the optimal value function

$$J_s(x) = \max_{(u_s,\dots,u_T)} \sum_{t=s}^{T} f(t, x_t, u_t) \quad \text{subject to} \begin{cases} x_s = x \\ x_{t+1} = g(t, x_t, u_t) \text{ for } t \geq s \\ u_t \in U \end{cases}$$

Bellman equation:

$$J_s(x) = \max_{u \in U} \left\{ f(s, x, u) + J_{s+1}(g(s, x, u)) \right\} \quad \text{for } s = 0,1,2,\dots,T-1$$

$$J_T(x) = \max_{u \in U} f(T, x, u)$$

Ex T:

$$\max \sum_{t=0}^{3} 1+x_t-u_t^2) \qquad x_{t+1}=x_t+u_t, \quad x_0=0, \quad U=\mathbb{R}$$

$$J_3(x) = \max_{u\in\mathbb{R}} \left(1+x-u^2\right) = 1+x \quad \text{at } u_3=0$$

$$J_2(x) = \max_{u_2\in\mathbb{R}} \left(1+x-u_2^2\right) + J_3(x+u_2) = \max_u \left\{1+x-u^2+1+x+u\right\}$$

$$' = -2u+1 \qquad u = 1/2$$
$$'' = -2 \qquad \text{ok.}$$

$$= 2+2x+1/4 = \underline{9/4+2x}, \quad u_2 = 1/2$$

$$J_1(x) = \max \quad 1+x-u^2+J_2(x+u) = \max \quad 1+x-u^2+9/4+2x+2u$$

$$' = -2u+2 \qquad u=1$$
$$'' = -2$$

$$= 1+x-1+9/4+2x+2$$

$$= \underline{17/4+3x}, \quad u_1 = 1$$

$$J_0(x) = \max \quad 1+x-u^2+17/4+3(x+u)$$

$$' = -2u+3 = 0 \qquad u=3/2$$
$$'' = -2$$

$$= 1+x-9/4+17/4+3x+9/2 = \underline{4x+15/2} \qquad u_0 = 3/2$$

With $x_0=0$:

$u_0 = 3/2$	$x_0 = 0$	$J_0(x) = \underline{4x+15/2} = \underline{\underline{15/2}}$
$u_1 = 1$	$x_1 = 3/2$	
$u_2 = 1/2$	$x_2 = 5/2$	
$u_3 = 0$	$x_3 = 3$	

Alternative solution methods

B) Euler equation

$$\max \sum_{t=0}^{T} F(t, x_t, x_{t+1}) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_1, \ldots, x_{T+1} \in U = \mathbb{R} \text{ free} \end{cases}$$

- special case: $x_{t+1} = u_t$, $g(t, x_t, u_t) = u_t$, $U = \mathbb{R}$
- many problems are covered by this via transformations

Euler equation:

If $x_0^*, x_1^*, \ldots, x_{T+1}^*$ is optimal solution, then it satisfies the difference equations

$$\boxed{\begin{aligned} F_2'(t, x_t, x_{t+1}) + F_3'(t-1, x_{t-1}, x_t) &= 0 \qquad \text{for } t = 1, 2, \ldots, T \\ F_3'(t-1, x_{t-1}, x_t) &= 0 \qquad t = T+1 \end{aligned}}$$

where F_2', F_3' are the partial derivatives of F with respect to variable 2 and 3.

Ex: $\max \sum_{t=0}^{T-1} \ln(x_t - \beta x_{t+1}) + \ln x_T$

$$\begin{cases} F_2'(t, x_t, x_{t+1}) + F_3'(t-1, x_{t-1}, x_t) = 0 \qquad t = 1, \ldots, T \\ \dfrac{1}{x_t - \beta x_{t+1}} - \dfrac{\beta}{x_{t-1} - \beta x_t} = 0 \\ x_{t-1} - \beta x_t = \beta(x_t - \beta x_{t+1}) \\ \underline{\beta^2 x_{t+1} - 2\beta x_t + x_{t-1} = 0} \qquad \text{(second order difference eqn.)} \end{cases}$$

$$\begin{cases} F_3'(T, x_T, x_{T+1}) = 0 \\ 0 = 0 \\ \text{(no condition)} \end{cases} \qquad (\text{since } F(T, x_T, x_{T+1}) = \ln(x_T))$$

c) Maximum principle and Hamiltonian

$$\max \sum_{t=0}^{T} f(t, x_t, u_t) \quad \text{s.t.} \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(t, x_t, u_t) \\ x_T \text{ free} \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

Hamiltonian:

$$H = \begin{cases} f(t,x,u) + p\,g(t,x,u) &, t < T \\ f(t,x,u) &, t = T \end{cases}$$

Maximum principle (necessary conditions)

If (x_t^*, u_t^*) is optimal, then there is a sequence p_t, $t = 0, 1, \ldots, T$ with

A) $H'_u(t, x_t^*, u_t^*, p_t) \cdot (u - u_t^*) \leq 0$ for all $u \in U$

B) $p_{t-1} = H'_x(t, x_t^*, u_t^*, p_t)$, $t = 1, 2, \ldots, T$

C) $p_T = 0$

Sufficient conditions

Suppose that (x_t^*, u_t^*) satisfy the conditions above, and that in addition

$H(t, x, u, p)$ is concave in (x, u) for all t

Then (x_t^*, u_t^*) is optimal.

(2) Infinite Horizon dynamic programming ← This will be explained in lecture 10.

$$\max \sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \quad \text{subject to} \quad \begin{cases} x_0 \text{ given} \\ x_{t+1} = g(x_t, u_t) \\ u_t \in U \subseteq \mathbb{R} \end{cases}$$

<u>Note</u>:
* instead of $t \leq T$ we have infinite horizon $t \to \infty$
* $\beta^t f(x_t, u_t) = f(t, x_t, u_t)$; $g(x_t, u_t) = g(t, x_t, u_t)$
  ($0 < \beta < 1$ is the one-periode discount factor)
* we assume that $f(x_t, u_t)$ is bounded, i.e. $|f(x_t, u_t)| < M$ for all t for some number $M > 0$. This implies that the sum
  $$\sum_{t=0}^{\infty} \beta^t f(x_t, u_t) \leq \sum_{t=0}^{\infty} \beta^t M = \frac{M}{1-\beta} \quad \text{is finite}$$

<u>Bellman equation</u>: Let $J(x) = J_0(x) = \max_{(u_t)} \sum_{t=0}^{\infty} \beta^t f(x_t, u_t)$ subj to ... $\begin{cases} x_0 = x \\ \vdots \end{cases}$

$$\boxed{J(x) = \max_{u \in U} \left\{ f(x,u) + \beta J(g(x,u)) \right\}}$$

- Functional equations: we want to solve for the function $J(x)$. Difficult to find max when $J(x)$ is not known, hence difficult to find $J(x)$ by solving the Bellman equation.

- When $0 < \beta < 1$ and $|f(x_t, u_t)| < M$, the Bellman equation has a unique bounded solution $J^*(x)$. If we "guess" $\mathcal{J}(x)$ and it fits in the equation, it is therefore the unique solution.

<u>Ex:</u>
Problem 12.3.1 in [FMEA]. See also (SM) Student Manual (online)