

# LECTURE 6

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Plan:

Constrained optimization

(Lagrange / Kuhn-Tucker)

DRE 7017

MATHEMATICS

Reading:

[AMEA] 3.3 - 3.6

[ME] 18 - 19

([S] 5-6, 7.7, 8.8)

# LECTURE 6

DRE 7007

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AUG 31 2012

MATHEMATICS

## PLAN:

- ① CONSTRAINED OPTIMIZATION
- LAGRANGE - KKT/TUCKER

## Reading:

- [FHEA] 3.3-3.6
- [ME] 18-19
- [S] 5-6, 7.7, 8.8

Let  $f: D \rightarrow \mathbb{R}$  be a function, defined on  $D \subseteq \mathbb{R}^n$ .

Consider

$$\max_{x \in D} f(x)$$

A point  $x^* \in D$  is a constrained maximum if  $x^* \in D$  is a maximizer such that  $x^* \in \partial D$ .

## Examples:

i)  $\min xyz$  when  $\begin{cases} x-y+2z=3 \\ x+y=3 \end{cases}$

$$f(x,y,z) = xyz, \quad D = \left\{ (x,y,z) : \begin{cases} x-y+2z=3 \\ x+y=3 \end{cases} \right\}$$

In this case,  $\partial D = D$ .

ii)  $\min xyz$  when  $\begin{cases} x-y+2z \geq 3 \\ x+y \geq 3 \end{cases}$

$$D = \underbrace{(D \cap \partial D)}_{\text{boundary}} \cup \underbrace{(D \setminus \partial D)}_{\text{interior}}$$

## Case I: Equality constraints.

### Standard form

$$\begin{array}{l} \max \\ \min \end{array} f(x_1, \dots, x_n) \text{ subj. to. } \begin{cases} g_1(\underline{x}) = b_1 \\ g_2(\underline{x}) = b_2 \\ \vdots \\ g_k(\underline{x}) = b_k \end{cases} \\ (k < n)$$

idea:  $\dim D$  is  $n-k$

### Non-degenerate constraint qualification (NDCQ)

$$\text{rk} \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1} & \frac{\partial g_k}{\partial x_2} & \dots & \dots \end{pmatrix} = k$$

Are there points  $x \in D$  such that NDCQ is not satisfied at  $x$ , i.e.,

$$\text{rk} \begin{pmatrix} \frac{\partial g_i}{\partial x_j}(x) \end{pmatrix} < k \quad ?$$

## Lagrange method:

Consider  $\max/\min f(\underline{x})$  subj. to  $\begin{cases} g_1(\underline{x}) = b_1 \\ \vdots \\ g_k(\underline{x}) = b_k \end{cases}$

Thm:

If  $\underline{x}^*$  is a solution to the max/min problem and if NDCQ is satisfied at  $\underline{x}^*$ , then the following condition holds:

(LC) If  $L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \dots - \lambda_k g_k(\underline{x})$ ,  
then for some  $\underline{\lambda}^* = \lambda_1^*, \dots, \lambda_k^*$ .

Foc  $\rightarrow$

$$\begin{aligned} \frac{\partial L}{\partial x_1}(\underline{x}^*, \underline{\lambda}^*) &= 0 \\ \frac{\partial L}{\partial x_2}(\underline{x}^*, \underline{\lambda}^*) &= 0 \\ \vdots \\ \frac{\partial L}{\partial x_n}(\underline{x}^*, \underline{\lambda}^*) &= 0 \end{aligned}$$

(Constraints)

## Cook book:

- i) Find all points  $(\underline{x}^*, \underline{\lambda}^*)$  that satisfies FOC + C.
- ii) Find all points  $\underline{x}^* \in D$  that does not satisfy NDCQ.

If there is a global max/min, then it must be one of the points in i) or ii)

To find out if there is a solution (global max/min),

- try
- a) Weierstr. thm
  - b) AD HOC.
  - c) Concavity/Convexity

## Sufficient conditions using convexity:

Assume that  $(\underline{x}^*, \lambda^*)$  that satisfy FOC + C.

Consider the function

$L(\underline{x}, \lambda^*)$  as a function in  $(x_1, \dots, x_n) = \underline{x}$

Then:

$D^2 L(\underline{x}, \lambda^*)$  <sup>negative semidef</sup>  $\leftarrow$  concave  $\Rightarrow \underline{x}^*$  is Max.  
 $D^2 L(\underline{x}, \lambda^*)$  <sup>positive semidef</sup> convex  $\Rightarrow \underline{x}^*$  is min.

## Interpretation of $\lambda$ :

Given  $\underline{b} = (b_1, b_2, \dots, b_k)$ , let  $\underline{x}^*(\underline{b})$  be a maximizer (minimizer) and let  $f^*(\underline{b}) = f(\underline{x}^*(\underline{b}))$  be the value at the max (min) for the given  $\underline{b}$ .

If  $\underline{x}^*(\underline{b})$  satisfy FOC when  $\lambda = \lambda^*$ , then

$$\lambda_i^* = \frac{\partial f^*(\underline{b})}{\partial b_i}$$

" $\lambda_i^*$  is the rate of change of the value function  $f^*(\underline{b})$  when  $b_i$  is changed"

i.e.  $\lambda_i^* > 0$  means that  $f(\underline{b})$  increases when  $b_i$  increases  
 $\lambda_i^* < 0$  ——— decreases when  $b_i$  ———

2) Case:

Inequality constraints

max  $f(x)$  subject to

$$\begin{cases} g_1(x) \leq b_1 \\ g_2(x) \leq b_2 \\ \vdots \\ g_k(x) \leq b_k \end{cases}$$

Standard form used in Kuhn-Tucker formulation.

A constraint  $g_i(x) \leq b_i$  is

binding if  $g_i(x) = b_i$

non-binding if  $g_i(x) < b_i$

Then:

If  $x^*$  is a solution to a standard K-T problem, and if  $x^*$  satisfy the NDCQ, then the following conditions hold:

$$\text{If } L(x, \lambda) = f(x) - \lambda_1 g_1(x) - \lambda_2 g_2(x) \dots - \lambda_k g_k(x)$$

then we have:

FOC

$$\begin{aligned} \frac{\partial L}{\partial x_1}(x^*, \lambda^*) &= 0 \\ \vdots \\ \frac{\partial L}{\partial x_n}(x^*, \lambda^*) &= 0 \end{aligned} \quad \text{for some } \lambda^*$$

and

CSC

$$\begin{aligned} \lambda_1^* &\geq 0 \quad \text{and} \quad \lambda_1^* \cdot (g_1^*(x^*) - b_1) = 0 \\ \lambda_2^* &\geq 0 \quad \text{and} \quad \lambda_2^* \cdot (g_2^*(x^*) - b_2) = 0 \\ \vdots & \\ \lambda_k^* &\geq 0 \quad \text{and} \quad \lambda_k^* \cdot (g_k^*(x^*) - b_k) = 0 \end{aligned}$$

$\Leftrightarrow$

For  $i=1, 2, \dots, k$ , we have

$\lambda_i \geq 0$  and

$\lambda_i = 0$  if  $g_i(x) < b_i$  is not bind.

## NDCQ in Kuhn-Tucker case:

Let  $\underline{x} \in D = \{ \underline{x} : g_1(\underline{x}) \leq b_1, \dots, g_k(\underline{x}) \leq b_k \}$ , and set

$$B = \{ i \in \{1, 2, \dots, k\} : g_i(\underline{x}) = b_i \}$$

= the set of indices corresponding to binding constraints at the given point  $\underline{x}$ .

Then the NDCQ for  $\underline{x} \in D$  is

$$\text{rk} \left( \frac{\partial g_i}{\partial x_j}(\underline{x}) : i \in B \right) = \#B$$

↑  
take only those rows where the constraint is binding at  $\underline{x}$

↑  
the number of elements in  $B$   
= the number of rows in the matrix

Easiest to check NDCQ when we decide in the different cases (according to binding/non-binding).

$k=2$ :

$$\left. \begin{array}{l} g_1 = b_1 \\ g_2 = b_2 \end{array} \right\} \text{rk} \left( \begin{array}{l} \frac{\partial g_1}{\partial x_j} \\ \frac{\partial g_2}{\partial x_j} \end{array} \right) = 2$$

$$\left. \begin{array}{l} g_1 = b_1 \\ g_2 < b_2 \end{array} \right\} \text{rk} \left( \frac{\partial g_1}{\partial x_j} \right) = 1$$

$$\left. \begin{array}{l} g_1 < b_1 \\ g_2 = b_2 \end{array} \right\} \text{rk} \left( \frac{\partial g_2}{\partial x_j} \right) = 1$$

$$\left. \begin{array}{l} g_1 < b_1 \\ g_2 < b_2 \end{array} \right\} \text{no condition}$$

Sufficient condition using concavity  
in the Kuhn-Tucker case:

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If  $(\underline{x}^*, \underline{\lambda}^*)$  satisfies FOC, CSC, C,  
then we consider

$L(\underline{x}, \underline{\lambda}^*)$  function in  $(x_1, \dots, x_n)$

If  $L(\underline{x}, \underline{\lambda}^*)$  is concave, then  $\underline{x}^*$  is  
a max.