LECTURE 5

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MATHEMATICS

Plan:

1. Optimization: Intro
2. Unconstrained optimization

Reading:

[FMEA] 3.1 - 3.2

[ME] 17

(LS] 2, 4, 7.3-7.6, 8.4-8.7

Problems:

Problem Set 5

1. Optimization

- Let $f: D \to \mathbb{R}$ be a function, with $D \subseteq \mathbb{R}^n$. We consider the problems

$$\max_{x\in D} f(x) = \max \{f: D \to \mathbb{R}\} = \max f(x) \text{ subj. to } x \in D$$

$$\min_{x\in D} f(x) = \min \{f: D \to \mathbb{R}\} = \min f(x) \text{ subj. to } x \in D$$

A solution to the max problem is an $x^* \in D$ s.t. $f(x^*) \geq f(x)$ for all $x \in D$. Then x^* is also called a maximizer. Similar for min problems.

Fact:

$\min_{x\in D} f(x)$ and $\max_{x\in D} -f(x)$ have the same solutions

We write $\arg\max \{f(x): x \in D\}$ for the set of maximizers. This set can be empty or have several points.

The attainable values of $\max_{x\in D} f(x)$ is the set $\{f(x): x\in D\} = f(D) \subseteq \mathbb{R}$.

The max problem has a solution if and only if $\sup f(D) \in f(D)$.

D: constraint set
f: objective function

Fact: If $\varphi: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function (i.e. such that $x < y \Rightarrow \varphi(x) < \varphi(y)$, then the max problems

$\max_{x\in D} f(x)$ and $\max_{x\in D} \varphi(f(x))$

have the same solution.

Ex: $\boxed{\max\ x^2+y^2 \text{ subj. to } 2x^2+3y^2 \le 6}$

and

$\boxed{\max\ \sqrt{x^2+y^2} \text{ subj. to } 2x^2+3y^2 \le 6}$

Same solution since

$x \mapsto \sqrt{x}$

is strictly increasing.

First problem is simpler.

Existence of solutions:

f cont., D compact $\overset{[W.]}{\Longrightarrow}$ at least one solution of $\boxed{\max\limits_{x \in D} f(x)}$

f not cont. or D not compact $\Longrightarrow$ the problem may or may not have solutions

② Unconstrained optimization

Let $\boxed{\max\limits_{x\in D} f(x)}$ be a max problem. If x^* is a maximizer, it is called unconstrained if $x^* \in D$ is an interior point of D. (i.e. not a boundary point)

$\boxed{x^* \in D \text{ interior point} \iff x^* \notin \partial D \ (x^* \in D \setminus \partial D)}$

Case 1: $D = \mathbb{R}^n$ or $D \subseteq \mathbb{R}^n$ open $\Longrightarrow$ all maximizers are unconstrained

Case 2: $D \subsetneq \mathbb{R}$ not open $\Longrightarrow$ there may be
- unconstrained maximizers (interior)
- constrained maximizers (boundary)

Defn: A point $x \in D$ is a local maximum of f on D if $f(x) \ge f(y)$ for all $y \in D \cap B(x,r)$ for some small $r>0$ (ie for all $y \in D$ close to x). Similar for local minimum.

Note: $\boxed{\text{maximum} \Longrightarrow \text{local maximum}}$

Defn: $x \in D$ is called stationary point if $Df(x) = 0$, i.e. if $\partial f/\partial x_1 = \dots = \partial f/\partial x_n = 0$, and a critical point if x is either stationary or $Df(x)$ does not exist (f not differentiable at x).

First order condition:

x^* unconstrained local max (or min) $\Rightarrow$ x^* is a critical point

The important thing is if $Df(x)$ exists, not that f is C^1 at x.

Either $Df(x^*) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}(x^*) = \dots = \frac{\partial f}{\partial x_n}(x^*) = 0$

or $Df(x^*)$ does not exist

Second order conditions:

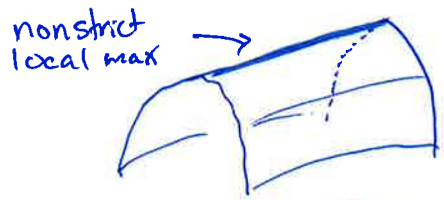
If f is C^2 on D and $x \in D \setminus \partial D$ interior pt. of D then

i) x local max (min) $\Rightarrow$ $D^2 f(x)$ negative semidefinite (positive)

Implication, not equivalence in i)

ii) $Df(x) = 0$ and $D^2 f(x)$ negative definite (positive) $\Rightarrow$ x strict local max (min)

local max but not strict local max looks like this



corresponds to $D^2 f(x)$ negative semidefinite but not negative definite

Summary:

If $\underline{x}^*$ is a stationary pt. for f, then

i) $D^2 f(\underline{x}^*)$ negative definite $\Rightarrow$ $\underline{x}^*$ strict local max

ii) $D^2 f(\underline{x}^*)$ positive definite $\Rightarrow$ $\underline{x}^*$ strict local min

iii) $D^2 f(\underline{x}^*)$ indefinite $\Rightarrow$ $\underline{x}^*$ saddle point (not local max or min)

Concave functions:

If f is concave, then any stationary pt. is a global maximizer.

Moreover, $\operatorname{argmax}\{f(x) : x \in D\}$ is empty (no max.)
or a convex set

If f is strictly concave, then $\operatorname{argmax}\{f(x) : x \in D\}$ is empty or a point.

If f is strictly quasi-concave, then any local max. is also a global maximizer. Moreover,

$$\operatorname{argmax}\{f(x) : x \in D\}$$

is either empty or a point.

See [S] Chap 6-7 for details.