LECTURE 4

DKE 7017

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MATHEMATICS

Plan:

1. Convex sets
2. Separation results
3. Convex / concave functions

Reading:

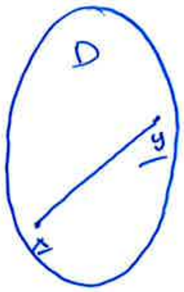
[FMEA] 2.2-2.3, 13.5-13.6
[ME] 21.1-21.2
([LS] 1.2, 1.6, 7)

Problems:

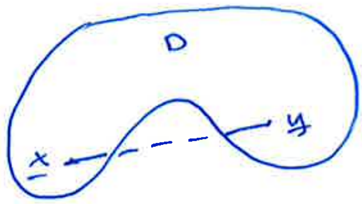
Problem Set 4

① Convex sets

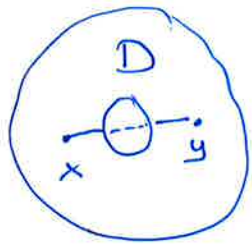
A subset $D \subseteq \mathbb{R}^n$ is <u>convex</u> if for any $\underline{x}, \underline{y} \in D$, the straight line segment $[\underline{x}, \underline{y}] = \{\underline{z} \in \mathbb{R}^n : \underline{z} = \lambda \cdot \underline{x} + (1-\lambda) \cdot \underline{y},\ \lambda \in [0,1]\} \subseteq D$.



convex set



<u>not</u> convex set



not convex

<u>Facts</u>:

i) If S, S' are convex sets in $\mathbb{R}^n$, then

$$S + S' = \{ s + s' : s \in S, s' \in S' \} \subseteq \mathbb{R}^n$$

is convex.

ii) If $(S_i)_i$ is a collection of convex sets, then

$\bigcap_i S_i$ is convex.

② Separation theorems

Let $\underline{p} \neq \underline{0}$ be a vector in $\mathbb{R}^n$. Then the set

$$H = \{ \underline{x} \in \mathbb{R}^n : \underline{p} \cdot \underline{x} = a \} \subseteq \mathbb{R}^n$$

is called a <u>hyperplane</u>, and we write $H = H(\underline{p}, a)$.

Two sets $D, \mathcal{E} \subseteq \mathbb{R}^n$ are <u>separated</u> by the hyperplane $H(\underline{p}; a) = H$ if $D, \mathcal{E}$ lie on opposite sides of H; i.e.

$\underline{p} \cdot \underline{y} \geq a$ for all $y \in D$

$\underline{p} \cdot \underline{y} \leq a$ — " — $y \in \mathcal{E}$

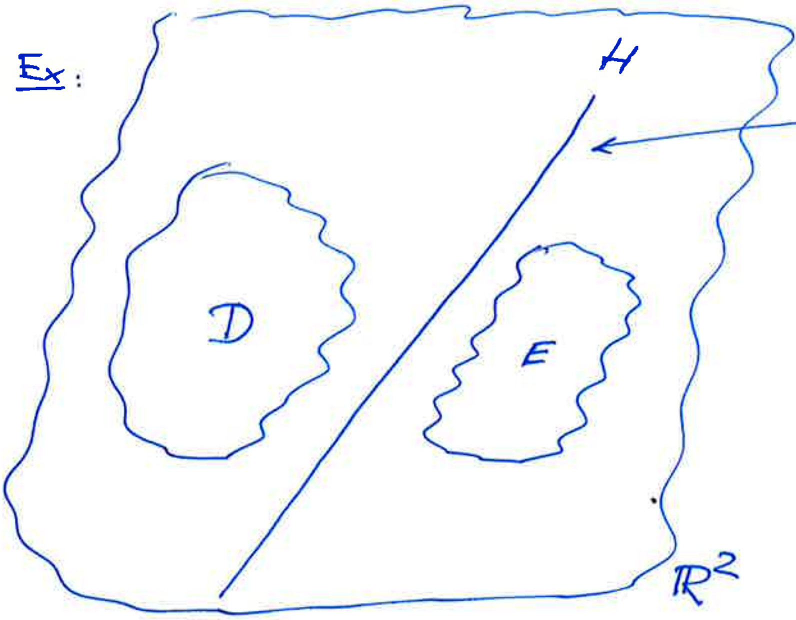
(or the other way around).

Notice that a hyperplane has equation

$$\underline{p} \cdot \underline{x} = a \iff p_1 x_1 + p_2 x_2 + \dots + p_n x_n = a$$

This a linear equation. If $n=2$, a hyperplane is a <u>line</u> and if $n=3$, a hyperplane is a <u>plane</u>.

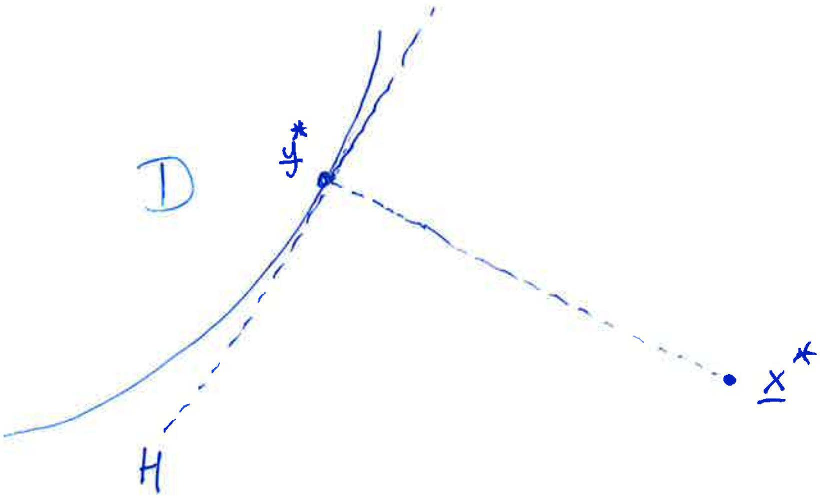
<u>Ex</u>:



hyperplane that separates D, E.

<u>Thm</u>:

If $D \subseteq \mathbb{R}^n$ is non-empty and convex set, and $\underline{x}^* \in \mathbb{R}^n \setminus D$, then there is a hyperplane $H = H(\underline{p}, a)$ that separates D and $\{\underline{x}^*\}$, with $\underline{p} \neq \underline{0}$. It is possible to choose $\|\underline{p}\| = 1$.



$\underline{y}^* \in D$ s.t. $d(\underline{x}^*, \underline{y}^*)$ is minimal

H: $\underline{p} \cdot \underline{x} = a$ with

$$\underline{p} = \underline{y}^* - \underline{x}^*$$
$$a = \underline{p} \cdot \underline{y}^*$$

Thm:
If $D, \mathcal{E} \subseteq \mathbb{R}^n$ are convex s.t. $D \cap \mathcal{E} = \emptyset$, then there is a hyperplane $H = H(\underline{p}, a)$ with $\underline{p} \neq \underline{0}$ that separates $D, \mathcal{E}$. We may choose $\|p\| = 1$.

Proof:
Let $\mathcal{F} = D + (-\mathcal{E})$, which is convex. Since $D \cap \mathcal{E} = \emptyset$, $0 \notin \mathcal{F}$.
Since $\underline{p} \cdot \underline{z} \geq \underline{p} \cdot \underline{0} = 0$ for all $\underline{z} = \underline{x} - \underline{y} \in \mathcal{F}$, it follows that we have $\underline{p} \cdot x \geq \underline{p} \cdot y$ for all $\underline{x} \in D, \underline{y} \in \mathcal{E}$. $\square$

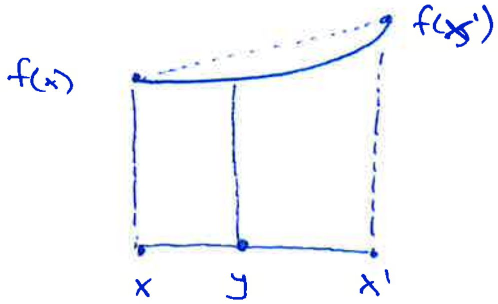
③ Convex / concave functions

Let $f: D \to \mathbb{R}$ be a function defined on a convex set $D \subseteq \mathbb{R}^n$.
Then f is convex if the following condition holds:

For any $x, x' \in D$ and any $y = \lambda \cdot x + (1-\lambda) \cdot x' \in [x, x']$ (with $\lambda \in [0,1]$), we have

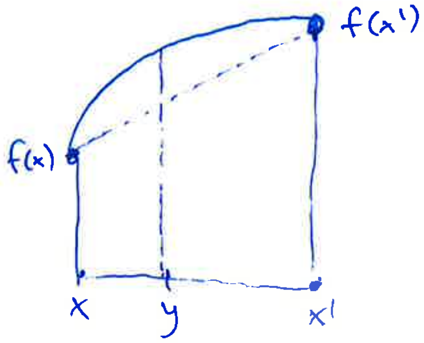
$$f(y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(x') \in [f(x), f(x')]$$

"the straight line segment from $(x, f(x))$ to $(x', f(x'))$ lies over (or on) the graph of f"



Strict concave / convex if $<$ resp. $>$

f is concave if $-f$ is convex, ie.



"the straight line segment from $(x, f(x))$ to $(x', f(x'))$ lies under (or on) the graph of f"

Thm:

Let f be a C^2 function on a subset $D \subseteq \mathbb{R}^n$ that is open and convex. Then we have

f concave $\Leftrightarrow$ $D^2 f(\underline{x})$ negative semidefinite for all $\underline{x} \in D$

f convex $\Leftrightarrow$ $D^2 f(\underline{x})$ positive ―――― " ――――

Similarly strict concave/convex correspond to pos./neg. definite.

Facts:

If $f: D \to \mathbb{R}$ is convex or concave on an open convex set $D \subseteq \mathbb{R}^n$, then f is continuous. Moreover, is differentiable for almost all $x \in D$, and C^1 in the points where it is differentiable.

Quasi-convex and quasi-concave functions

Let $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is a convex set.

We say that f is

<u>quasi-concave</u> if $U_f(a) = \{x \in D : f(x) \geq a\}$ is a convex set for all $a \in \mathbb{R}$

<u>quasi-convex</u> if $L_f(a) = \{x \in D : f(x) \leq a\}$ is a convex set for all $a \in \mathbb{R}$

convex $\Rightarrow$ quasi-convex
concave $\Rightarrow$ quasi-concave

<u>Ex:</u> $f(x,y) = x^2 + y^2$ defined on $\mathbb{R}^2$ is a typical convex fn.

$U_f(a) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq a\}$ is convex.


$x^2 + y^2 \leq a$