

LECTURE 3

EIVIND ERIKSEN

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DRE 7017

MATHEMATICS

Plan:

- ① Functions and continuity
- ② Derivatives and partial derivation

Reading:

[FHEA] 13.3

[NEJ] 13.4, 14

([S] 1.4)

Problems:

Problem set 3

① Functions and continuity

A function $f: X \rightarrow Y$ is a rule that assigns a unique value $y = f(x) \in Y$ to every element $x \in X$. The set X is called the domain and Y is called the codomain of f .

Most functions we consider are functions $f: D \rightarrow \mathbb{R}$, where the domain $D \subseteq \mathbb{R}^n$ and the codomain is \mathbb{R} .

Example: $f(x,y) = e^{xy} - 1$, which can be written

$$f: \mathbb{R}^2 \rightarrow \mathbb{R} \\ (x,y) \mapsto e^{xy} - 1$$

$g(x,y) = \frac{1}{x^2+y^2}$, $(x,y) \neq (0,0)$ which
can be written

$$g: D \rightarrow \mathbb{R} \quad \text{with } D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\} \\ (x,y) \mapsto \frac{1}{x^2+y^2}$$

But also this is a function:

$$C([a,b], \mathbb{R}) \rightarrow \mathbb{R} \\ f \mapsto \int_a^b f(x) dx$$

Such functions, where the domain consists of functions, are often called operators.

Let $f: X \rightarrow Y$ be a function, where X and Y are metric spaces. We say that f is continuous at $x \in X$ if the following condition holds:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that
 $x' \in X$ with $d(x', x) < \delta \Rightarrow d(f(x'), f(x)) < \varepsilon$

In other words, $x' \in B(x, \delta) \Rightarrow f(x') \in B(f(x), \varepsilon)$

~~we say that~~
We say that f is continuous if it is continuous at x for all $x \in X$.

Consequences of continuity:

Theorem:

If $f: X \rightarrow Y$ is continuous and $K \subseteq X$ is compact, then

$$f(K) = \{ y = f(x) : x \in K \} \subseteq Y$$

is also compact.

Theorem: (Weierstrass) (Extreme value thm.)

If $f: D \rightarrow \mathbb{R}$ is continuous, where $D \subseteq \mathbb{R}^n$ is a closed and bounded set, then f has a maximum and a minimum.

Proof:

D is compact, so $f(D)$ is compact. But the compact sets in \mathbb{R} are closed and bounded. Let $M = \sup f(D)$ and $m = \inf f(D)$. Since $f(D)$ is closed and bounded, there are x_{\max} and x_{\min} in D such that $f(x_{\max}) = M$, $f(x_{\min}) = m$.

How to determine if a function is continuous

Facts: * All "elementary" functions are continuous (polynomials, rationals, exponentials, logarithms).

* Sums, products and quotients of continuous functions are cont.

* Compositions of cont. functions are cont.

Ex:

$$f(x) = \begin{cases} x+1, & x \geq 0 \\ e^x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

Function spaces

Let $D \subseteq X$ be a subset of a metric space X . Define

$$C(D) = \{f: D \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$$

Facts:

i) $C(D)$ is a vector space

ii) The sup norm is a norm on $C(D)$, given by

$$\|f\|_{\text{sup}} = \sup_{x \in D} |f(x)| = \sup \{ |f(x)| : x \in D \}$$

for $f \in C(D)$.

iii) If D is compact, then $C(D)$ with sup norm is a complete metric space.

Note: Alternative defn. of continuity

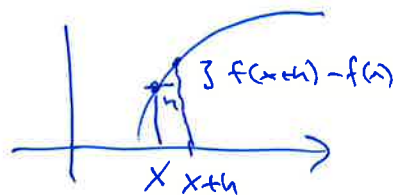
$f: X \rightarrow Y$ cont. at $x \in X$ if and only if the following condition holds:

For any sequence (x_i) in X with $\lim (x_i) = x$, we have $\lim (f(x_i)) = f(x)$.

② Derivatives and partial derivatives

Recall: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function in one variable and $x \in \mathbb{R}$, then

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$



Let $f: D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ is an open set. For any $\underline{x} \in D$, the partial derivatives

$$f'_i(\underline{x}) = \frac{\partial f}{\partial x_i}(\underline{x}) = \lim_{h \rightarrow 0} \frac{f(\underline{x} + \underline{e}_i h) - f(\underline{x})}{h}$$

, where $\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ (1 is in position i)

if this limit exists.

For any $\underline{x} \in D$, the total derivative is $Df(\underline{x})$ if there is a $1 \times n$ -matrix $Df(\underline{x}) = A$ such that the following condition holds:

For any $\varepsilon > 0$, there is a $\delta > 0$ such that
 $\underline{y} \in D, \|\underline{y} - \underline{x}\| < \delta \Rightarrow \|f(\underline{y}) - f(\underline{x}) - A \cdot (\underline{y} - \underline{x})\| < \varepsilon \cdot \|\underline{y} - \underline{x}\|$

In other words,

$$\lim_{\underline{y} \rightarrow \underline{x}} \frac{\|f(\underline{y}) - f(\underline{x}) - A(\underline{y} - \underline{x})\|}{\|\underline{y} - \underline{x}\|} = 0$$

$$A = \left(\frac{\partial f}{\partial x_1}(\underline{x}) \quad \frac{\partial f}{\partial x_2}(\underline{x}) \quad \dots \quad \frac{\partial f}{\partial x_n}(\underline{x}) \right)$$

If $Df(\underline{x})$ exists, we say that f is differentiable in \underline{x} . If it is differentiable for all $\underline{x} \in D$, f is called differentiable.

Facts:

i) If f is differentiable at $\underline{x} \in D$, then all partial derivatives $\frac{\partial f}{\partial x_i}(\underline{x})$ exists, and $Df(\underline{x}) = \left(\frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \dots, \frac{\partial f}{\partial x_n}(\underline{x}) \right)$.

ii) If all partial derivatives $\frac{\partial f}{\partial x_i}(\underline{x})$ exists and are continuous at \underline{x} , then f is differentiable and $Df(\underline{x}) = \left(\frac{\partial f}{\partial x_1}(\underline{x}) \quad \dots \quad \frac{\partial f}{\partial x_n}(\underline{x}) \right)$.

f is called a C^1 function if $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ exists and are continuous

Facts:

- i) Sums of differentiable functions are differentiable
- ii) Composition of differentiable functions are differentiable.

Assume that $f: D \rightarrow \mathbb{R}$ is a C^1 function. We denote the j 'th partial derivative of $(\partial f / \partial x_i): D \rightarrow \mathbb{R}$ by $\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{ij}$ if it exists. We say that f is twice differentiable at x , with second derivative

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

In that case. The matrix $D^2 f(x)$ is also called the Hessian of f .

We say that f is C^2 if $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists and is continuous for all ij .

Theorem:

If $f: D \rightarrow \mathbb{R}$ is C^2 , then the Hessian $D^2 f(x)$ is a symmetric matrix.

Ex:

Determine the definiteness of $D^2 f(x, y)$ for all pts $(x, y) \in \mathbb{R}^2$ when

$$f(x, y) = e^{xy} - 1.$$