LECTURE I

DKE 2017

MATHEMATICS

EIVIND ERIKSEN

AUG 18TH '14

Lecture plan:

1. Matrices, linear systems, Gaussian elimination
2. Eigenvectors, eigenvalues and diagonalization
3. Quadratic forms, definiteness of symmetric matrices

Reading:

[FMEA] 1.1-1.7

[ME] 6-7, 23

([S] 1.3, 1.5)

[LSGE]

Problems

Problem Set I

Lecture 2: Tuesday Aug 19th at 10-12 in A2-030

(tomorrow)

on Euclidean spaces. Sequences. Topology

(see course page / It's L. for full lecture plan).

① Matrices and matrix algebra

An $m\times n$-matrix A is a rectangular array (with m rows and n columns) of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

A (column) vector is an $m\times 1$-matrix

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Operations on matrices:

* Addition / subtraction: $\begin{cases} A+B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \\ A-B = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij}) \end{cases}$

  - defined when A, B has same size
  - computed position by position

* Scalar multiplication: $r \cdot A = r \cdot (a_{ij}) = (r a_{ij})$

  - defined when r is scalar (number)
  - computed position by position

* Multiplication: $\underset{(m\times n)}{A} \cdot \underset{(n\times p)}{B} = (a_{ij}) \cdot (b_{ij}) = (c_{ij})$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

  - defined when #cols(A) = #rows(B)
  - not commutative: $AB \neq BA$

* Transpose: $\underset{(m\times n)}{A} \rightsquigarrow A^T = (a_{ij})^T = \underset{(n\times m)}{(a_{ji})}$

Special matrices:

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$ zero matrix (any size)

$A \cdot 0 = 0$
$0 \cdot A = 0$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$ identity matrix $(n \times n)$

Property: $\boxed{A \cdot I = A \quad I \cdot A = A}$

Square matrix: #rows = #cols

Diagonal: $A = (a_{ij})$ square such that $\{ a_{ij} = 0$ for $i \neq j$
$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Symmetric: $A = (a_{ij})$ square such that $A^T = A$

Upper/lower triangular: $\begin{pmatrix} d_1 & & * \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix}$, $\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ * & & d_n \end{pmatrix}$

Inverse matrix: A $n \times n$-matrix

An inverse of A is a matrix B such that $A \cdot B = B \cdot A = I_n$

If it exists, it is unique and written A^{-1} (instead of B)

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\Longrightarrow$ $ad - bc = 0$: A^{-1} does not exist

2x2-matrix $\qquad ad - bc \neq 0$: $\boxed{A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Determinant: A $n \times n$-matrix $\rightsquigarrow$ $\det(A) = |A|$ is a number

Determinant $\det(A)$ defined inductively:

i) $n = 1$: $A = (a) \to |A| = a$

ii) General case: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \Rightarrow |A| = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} & \\ a_{32} & \ddots & \\ \vdots & & \ddots \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} & \cdots \\ a_{31} & a_{33} & \cdots \\ \vdots & & \ddots \end{vmatrix} + \ldots$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boxed{\det(A) = |A| = a \cdot d - bc}$

2x2

Property: A $n \times n$-matrix

A^{-1} exists $\Longleftrightarrow |A| \neq 0$

Linear systems and Gaussian elimination

A **linear system** is a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \quad\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix form:

$$A \cdot \underline{x} = \underline{b}$$

Augmented matrix:

$$(A \mid \underline{b})$$

Gaussian elimination is a solution method; see [LSGE] for details.

Example:

$$\begin{cases} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{cases} \Rightarrow$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right)$$

$$\downarrow$$

pivot positions

elementary row operations

Echelon form (zeros under all pivots)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\Downarrow$$

Linear system:

$$\begin{aligned} x + y + z &= 3 \quad \leftarrow x = 1 \\ y + 3z &= 4 \quad \leftarrow y = 1 \\ 2z &= 2 \quad \leftarrow z = 1 \end{aligned}$$

Solution: $x=1, y=1, z=1$

Linear systems and Gaussian elim.

<u>Ex:</u> Linear system

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

Gaussian elimination is an efficient method for solving linsys.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \xrightarrow[\text{operations}]{\text{elementary row}} \left(\begin{array}{c} \text{echelon} \\ \text{form} \end{array}\right)$$

coeff. matrix

augmented matrix

echelon form = only zeros under each leading coeff.

1. switch two rows
2. multiply a row with $c \neq 0$
3. add a multiple of one row to another row

leading coeff. = first non-zero coeff. in a row

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l}]\,\cdot(-1) \\ \\ \\ \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l}]\,\cdot(-1) \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ ]\,\cdot(-2) \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \qquad \begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ 2z &= 2 \end{aligned} \qquad \begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \end{aligned}$$

echelon form

back substitution

pivot, pivot position
= leading coeff. (pos.) in an echelon form.

Gauss-Jordan elim.:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) :2 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \\ -3 \\ -1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \end{aligned}$$

reduced echelon form
= echelon form with
i) all pivots are 1
ii) all entries over a pivot are 0

Important facts:

- Any matrix has an echelon / reduced echelon form
- An echelon form is not unique, but the pivot positions are.
- The reduced echelon form is unique.

geometric interpretation:

3x3 linear sys.
= intersection of 3 planes in 3-dim space

Ex:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x \quad +3z+4w &= 7 \\ z &= 2 \end{aligned}$$

$\underline{z=2}$

$x + 6 + 4w = 7 \Rightarrow \underline{x = 1 - 4w}$

y, w : free variables (non-pivot col's)

x, z : basic variables (pivot col's)

$$\left.\begin{aligned} x &= 1-4w \\ y &= y \\ z &= 2 \\ w &= w \end{aligned}\right\} \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1-4w \\ y \\ 2 \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + w \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Solution set is a <u>plane</u>.

\# free variables

= degrees of freedom

= dim. of the set of solutions

Ex:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

pivot in the last col.

$\Rightarrow$ no solutions

$(0 = 3)$

Rank

The rank of a matrix A is the number of pivot positions in its echelon form. It is written $\text{rk}\, A$.

Let $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be n m-vectors. The vectors are called linearly independent if $x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_n\underline{v}_n = \underline{0}$ has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$, and linearly dependent if there are non-trivial solutions.

Fact: When $A = (\underline{v}_1 \,|\, \underline{v}_2 \,|\, \cdots \,|\, \underline{v}_n)$, then $\text{rk}\, A$ is the maximal number of linearly independent vectors among $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$.

Let A be an $m\times n$-matrix. A $k\times k$-submatrix of A is a matrix B obtained by selecting k rows $(i_1, i_2, \ldots, i_k)$ and k columns $(j_1, j_2, \ldots, j_k)$ from A. A minor of order k from A is the determinant of a $k\times k$-submatrix.

Fact: $\text{rk}\, A$ is the maximal order k such that there exists a non-zero minor of order k;

$$\text{rk}\, A = \max\{k : M_k \neq 0 \text{ for a minor } M_k \text{ of order } k\}$$

Example:

$$A = \begin{pmatrix} 1 & 7 & 5 \\ 3 & 4 & -2 \end{pmatrix} \qquad \begin{vmatrix} 1 & 7 \\ 3 & 4 \end{vmatrix} = 4 - 21 = -17 \neq 0$$

$$\Rightarrow \text{rk}\, A = 2$$

A principal minor is a minor obtained by selecting k rows $i_1, i_2, \ldots, i_k$ and the same columns $j_1 = i_1, j_2 = i_2, \ldots, j_k = i_k$.

A leading principal minor is a minor obtained by selecting rows $1, 2, 3, \ldots, k$ and columns $1, 2, \ldots, k$.

Rank: $\operatorname{rk} A$ = # pivot positions in A.

If A is $n \times n$-matrix:

$\operatorname{rk} A = n \iff |A| \neq 0$
(maximal rank)

You can compute determinant / inverses efficiently using Gaussian elimination.

(2) Eigenvalues / eigenvectors

A : $n \times n$-matrix

λ is an eigenvalue if $A \cdot \underline{x} = \lambda \cdot \underline{x}$ has non-trivial solutions

linear system.

Linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$A = (a_{ij})$

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & a_{12} \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & b_2 \\ & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) \longleftrightarrow A\underline{x} = \underline{b}$$

augm. matrix

λ eigenvalue $\Leftrightarrow$ $|A - \lambda I| = 0$

char. eqn.

polynomial eqn. of order n

An eigenvector of A with eigenvalue λ is a solution of $A\underline{x} = \lambda \underline{x}$.

Ex: $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

$$\begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(-1-\lambda) - 9 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4\cdot(-16)}}{2}$$

$$= 3 \pm 5$$

$\underline{\lambda_1 = 8}$, $\lambda_2 = \underline{-2}$

eigenvalues

eigenvectors:

$\underline{\lambda = 8}$: $\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \underline{x} = \underline{0}$

$$\underline{x} = \underline{\underline{t \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}}}$$

$\underline{\lambda = -2}$: $\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \underline{x} = \underline{0}$

$$\underline{x} = \underline{\underline{t \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}}}$$

(2) Eigenvalues and eigenvectors

Let A be an $n \times n$-matrix.

A number λ is an <u>eigenvalue</u> for A if $A \cdot \underline{v} = \lambda \cdot \underline{v}$ has a non-trivial solution $\underline{v} \neq \underline{0}$. In that case, the <u>eigenspace</u> of <u>eigenvectors</u> of A with eigenvalue λ is

$$E_\lambda = \{ \underline{v} : A\underline{v} = \lambda \underline{v} \}$$

The equation $A\underline{v} = \lambda \underline{v}$ can be rewritten as $(A - \lambda I)\underline{v} = \underline{0}$.

<u>Fact:</u> The eigenvalues of A are the solutions of the <u>characteristic equation</u> $\det(A - \lambda I) = 0$. It is a polynomial equation of order n.

<u>Example:</u>

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \det(A - \lambda I) = \underbrace{\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}}_{A - \lambda I} = (1-\lambda)\cdot(1-\lambda) - 2^2 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\underline{\lambda = 3}, \ \underline{\lambda = -1}$$

<u>Fact:</u> The characteristic equation is $(-1)^n \cdot \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_{n-1}\lambda + c_n = 0$ where $c_1 = tr(A)$ and $c_n = \det(A)$

$$(tr\, A = a_{11} + a_{22} + \dots + a_{nn})$$

<u>Fact:</u> If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the n eigenvalues of A, then

- $$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$$
  $$tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$\det(A - \lambda I) =$

$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

<u>Fact:</u> If A is symmetric, then A has n real eigenvalues (counted with multiplicity).

If A is symmetric, then we have

$$\det(A-\lambda I) = (\lambda-\lambda_1)^{n_1}\cdot(\lambda-\lambda_2)^{n_2}\cdot\ldots\cdot(\lambda-\lambda_k)^{n_k}$$

with $n_1+n_2+\ldots+n_k=n$. The number n_i is the (algebraic) multiplicity of λ_i.

Fact:

If λ is an eigenvalue of A of multiplicity m, then the equation $(A-\lambda I)\cdot \underline{v}=\underline{0}$ has at most m degrees of freedom.

Example:

$$A=\begin{pmatrix}3&1\\0&3\end{pmatrix}$$

$$A=\begin{pmatrix}2&3\\3&2\end{pmatrix}$$

A square matrix A is called diagonalizable if there is diagonal matrix D and a invertible P such that $A=PDP^{-1}$.

Fact: If A has eigenvalues $\lambda_1,\ldots,\lambda_n$ and linearly independent eigenvector $\underline{v}_1,\ldots,\underline{v}_n$ then $A=PDP^{-1}$ if

$$D=\begin{pmatrix}\lambda_1&&&0\\&\lambda_2&&\\&&\ddots&\\0&&&\lambda_n\end{pmatrix},\quad P=\left(\underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \ldots \,\vdots\, \underline{v}_n\right)$$

If $\lambda_1,\lambda_2,\ldots,\lambda_k$ are k distinct eigenvalues for A, with multiplicities $m_1,m_2,\ldots,m_k$, then A is diagonalizable if and only if the following conditions hold:

i) $m_1+m_2+\ldots+m_k=n$

ii) The equation $(A-\lambda_i I)\underline{v}=\underline{0}$ has m_i degrees of freedom for $i=1,2,\ldots,k$.

Example:

$A = \begin{pmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{pmatrix}$ We want to compute A^n when $n \gg 0$

If A diagonalizable, then $A = PDP^{-1}$, and we have

inv. matrix diagonal matrix

$$A^n = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$$
$$= PD^nP^{-1}$$
$$= P \cdot \begin{pmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n & \\ 0 & & \ddots \; \lambda_k^n \end{pmatrix} P^{-1} \quad \text{if } D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_k \end{pmatrix}$$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}$

Eigenvalues: $\lambda = 1$, $\lambda = 0.75$

$\underline{v_1}$ $\underline{v_2}$

$P = (\underline{v_1} \mid \underline{v_2})$

$$A^n = P \cdot \begin{pmatrix} 1^n & 0 \\ 0 & 0.75^n \end{pmatrix} \cdot P^{-1} \xrightarrow[n \to \infty]{} P \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot P^{-1}$$

Diagonalización:

A diag. of $\underline{A}$ is $\begin{cases} D : \text{diagonal} \\ P : \text{invertible} \end{cases}$

such that $P^{-1} A \cdot P = D$.

Ex: $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

$$\boxed{\begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 7 & 3 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}}$$

$D = \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix}$ $\qquad P = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$

diagonal matrix of eigenvalues

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ $\qquad P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$

eig. vector for λ_1 $\qquad$ eig. vect. for λ_2

A is <u>diagonalizable</u> $\Leftrightarrow$ there are enough eigenvalues and eigenvectors

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda^2 + 1 = 0$

not <u>diag.</u>

Ex: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = 1 \quad \lambda_2 = 1$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 3 & \cancel{1} \\ 1 & -3 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \underline{x} = \underline{0}$$

$$\lambda = 1 \qquad \underline{x} = t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

one free var.

The vectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ are linearly independent if

$$c_1 \cdot \underline{v_1} + c_2 \underline{v_2} + \dots + c_n \underline{v_n} = \underline{0}$$

has only the trivial solution $c_1 = c_2 \dots = c_n = 0$.

$\Updownarrow$

no vector is a linear comb. of the others

To have a diagonalization of A, the column vectors of P (eigenvectors) have to be linearly independent.

$\boxed{\text{A diagonalizable} \iff \text{A has } n \text{ linearly independent eigenvectors}}$

Fact: A symmetric $\Longrightarrow$ A diagonalizable
$(A^T = A)$

(3) Definiteness

A quadratic form in n variables is a polynomial function where all terms have degree two,

$$Q(x_1, x_2, \dots, x_n) = a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{nn}x_n^2$$
$$\shortparallel$$
$$Q(\underline{x})$$

A quadratic form can be written in matrix form

$$Q(\underline{x}) = (x_1\ x_2 \dots x_n)\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{x}^T A \underline{x}$$

for a $\begin{cases} \text{matrix } A \text{, or} \\ \text{unique symmetric } n{\times}n\text{-matrix } A. \end{cases}$

Definition: Q and A are called

positive semidefinite $\iff$ $\underline{x}^T A \underline{x} \geq 0$ for all $\underline{x}$

negative —"— $\quad \underline{x}^T A \underline{x} \leq 0$ —"—

indefinite $\quad \underline{x}^T A \underline{x}$ have positive and negative values

positive definite $\iff$ $\underline{x}^T A \underline{x} > 0$ for all $\underline{x} \neq \underline{0}$

negative —"— $\quad \underline{x}^T A \underline{x} < 0$ —"—

Fact: If A is symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$, then we have

$\lambda_1, \lambda_2, \dots, \lambda_n \geq 0 \iff A$ positive semidefinite

> 0 positive definite

≤ 0 negative semidefinite

< 0 negative definite

$\lambda_i > 0,\ \lambda_j < 0$ indefinite

3 Definiteness of quadratic forms.

Quadratic form = polynomial where all terms have degree 2.

Ex: $Q(\underset{\sim}{x}) = x_1^2 + 2x_1x_2 - 3x_2^2$, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= (x_1 \; x_2) \cdot \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$Q(\underset{\sim}{x}) = \underset{\sim}{x}^T \cdot A \cdot \underset{\sim}{x}$ for an $n \times n$-matrix A

$\uparrow$
n var's

A can be chosen to be symmetric

coeff. in front of $x_i x_j$: $\begin{cases} a_{ij} + a_{ji} & i \neq j \\ a_{ii} & i = j \end{cases}$

Definiteness: of Q or of A

$Q(\underset{\sim}{x}) \geq 0$ for all $\underset{\sim}{x}$: Q positive semidef.

$Q(\underset{\sim}{x}) \leq 0$ for all $\underset{\sim}{x}$: negative — " —

none of the above : indefinite

$Q(x) > 0$ for all $\underset{\sim}{x} \neq \underset{\sim}{0}$: positive definite

$Q(x) < 0$ — " — : negative — " —

A symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$:

$\lambda_1, \dots, \lambda_n \geq 0$	pos. semidef.
$\lambda_1, \dots, \lambda \leq 0$	neg. — " —
all other cases	indefinite
$\lambda_1, \dots, \lambda_n > 0$	pos. definite
$\lambda_1 \dots \lambda_n < 0$	neg. — " —

Ex: $x_1^2 + 2x_1x_2 - 6x_2^2$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -6 \end{pmatrix}$$

$$\lambda^2 + 5\lambda - 7 = 0$$

$$\lambda_1 = \frac{-5 \pm \sqrt{25 + 28}}{2}$$

one pos., one neg. $\lambda \Rightarrow Q$ indefinite

Effective method:

Let A be symmetric $n \times n$-matrix, let $D_1, D_2, \dots, D_n$ be its leading principal minors, and let $\Delta_1, \Delta_2, \dots, \Delta_n$ be any of its principal minors

A positive definite $\iff D_1, D_2, \dots, D_n > 0$

A negative " $\quad D_1 < 0, D_2 > 0, D_3 < 0, \dots$ (that is, $(-1)^i D_i > 0$)

A positive semidefinite $\iff \Delta_1, \Delta_2, \dots, \Delta_n \geq 0$ for <u>all</u> principal minors

A negative " $\quad \Delta_1 \leq 0, \Delta_2 \geq 0, \dots$ — " —

Problems: Problem Set I