Lecture 7

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AUG 29, 2014

~~SEP 3RD 2012~~

DRE 7017

Mathematics

Plan:

1. Ordinary differential equations
2. Systems of differential equations
3. Linearizations

Reading:

[FMEA] 5-7
[ME] 24-25

① An ODE (ordinary differential equation) is an equation relating a function $y = y(t)$ and its derivative (and possibly higher order derivatives). A first order ODE typically has the form

$$\dot{y} = F(y,t)$$

where F is some function in (y,t). The variable t often is time. An ODE is autonomous if the expression for $\dot{y}$ does not depend on t; i.e. that

$$\dot{y} = F(y)$$

in the order one case.

Example: $\dot{y} = ay + b$, with $a, b \in \mathbb{R}$ constants.

Solution methods: a) Separation
b) Int. factor
c) Linear methods

Constant solution = steady state: $y = \bar{y}$ constant solution

$$\Updownarrow$$

$$a\bar{y} + b = 0$$

$$\Updownarrow$$

$$\bar{y} = -\frac{b}{a} \quad (a \neq 0)$$

Let $z = y - \bar{y}$. Then we have:

$z' = y'$; $ay + b = a(z + \bar{y}) + b = az + a(-\frac{b}{a}) + b = az + b - b = az$

Hence

$$y' = ay + b \iff z' = az \quad \text{with } z = y - \bar{y}$$

Solution:

$z' = az \Rightarrow z = Ce^{at} \Rightarrow y - \bar{y} = Ce^{at} \Rightarrow y = \bar{y} + Ce^{at} = -\frac{b}{a} + Ce^{at}$

Stability:

$a > 0$: $y = -\frac{b}{a} + Ce^{at} \to \infty$ as $t \to \infty$ — not stable

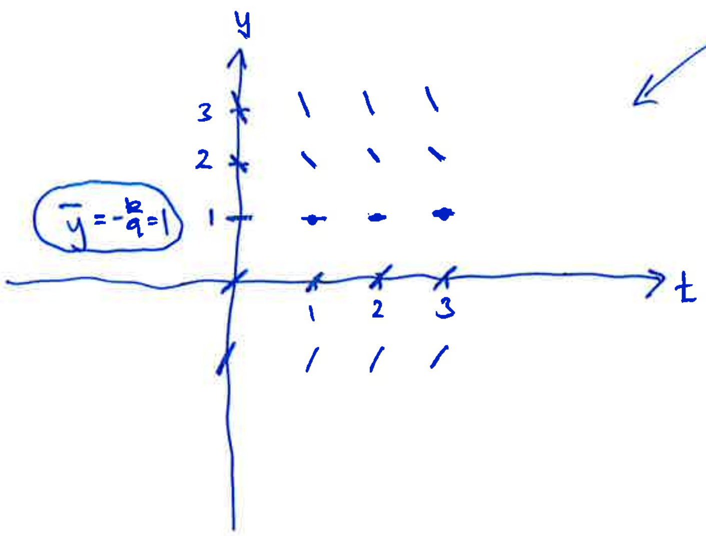
$a < 0$: $y = \bar{y} + Ce^{at} \to \bar{y}$ as $t \to \infty$ — (globally asymptotically) stable with equilibrium $\bar{y} = -\frac{b}{a}$

Note that $y_0 = y(0) = -\frac{b}{a} + C \Rightarrow C = y_0 + \frac{b}{a} = y_0 - \bar{y}$, so C is given by y_0.

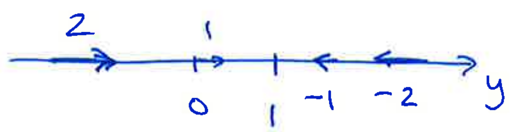
Phase diagram: Case $a = -1, b = 1$

$y = 1, t = 1 \Rightarrow y' = a \cdot y + b = 1 - y$

We draw a small line segment at (t, y) with slope $y' = 1 - y$.



or



We draw an arrow at (y) with length $|y'| = |1-y|$ and arrow $\to (+)$ or $\leftarrow (-)$.

(2) Linear system of ODE's

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + \dots + a_{1n}y_n + b_1 \\ &\vdots \\ y_n' &= a_{n1}y_1 + \dots + a_{nn}y_n + b_n \end{aligned}\right\} \iff \underline{y}' = A\underline{y} + \underline{b}$$

Steady state: $\bar{\underline{y}} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix} \in \mathbb{R}^n$ (constants) such that $A\bar{\underline{y}} + \underline{b} = \underline{0}$

$A\bar{\underline{y}} = -\underline{b}$

(linear system)

If $\bar{\underline{y}}$ is steady state, then $\underline{z} = \underline{y} - \bar{\underline{y}}$ transforms $\underline{y}' = A\underline{y} + \underline{b}$ into $\underline{z}' = A\underline{z}$.

Thm: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A and $\underline{v}_1, \dots, \underline{v}_n$ are corresponding eigenvectors that are linearly independent, then

$$\underline{z} = c_1 \cdot \underline{v}_1 \cdot e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + \dots + c_n \underline{v}_n e^{\lambda_n t}$$ is gen. solution to $\underline{z}' = A\underline{z}$

In particular, the general solution to the original $\underline{y}' = A\underline{y} + \underline{b}$ is

$$\underline{y} = c_1 \underline{v}_1 e^{\lambda_1 t} + \dots + c_n \underline{v}_n e^{\lambda_n t} + \bar{\underline{y}}$$

Ex: $$\begin{aligned} y_1' &= y_1 - 3y_2 + 2 \\ y_2' &= 2y_1 - 4y_2 + 2 \end{aligned} \qquad \begin{aligned} y_1 &= 1 + c_1 \cdot 3e^{-t} + c_2 \cdot e^{-2t} \\ y_2 &= 1 + c_1 \cdot 2e^{-t} + c_2 \cdot e^{-2t} \end{aligned}$$

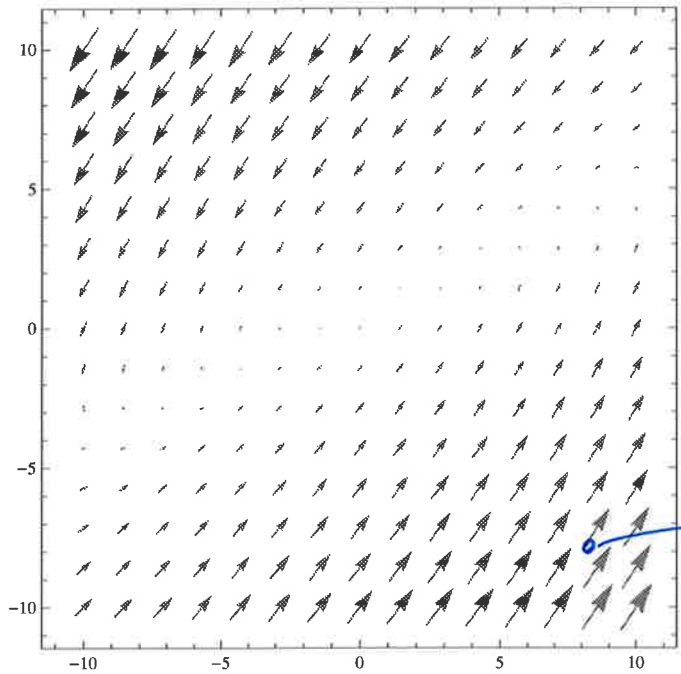
Pf. of thm:

$A\underline{z} = \underline{z}'$ with $\underline{z} = P\underline{u}$ gives diff. eqn. in new var's $\underline{u}$:

$$\left.\begin{aligned} \underline{z}' &= (P\underline{u})' = P\underline{u}' \\ A\underline{z} &= AP\underline{u} = PD\underline{u} \end{aligned}\right\} \quad \begin{aligned} P\underline{u}' &= PD\underline{u} \\ \underline{u}' &= D\underline{u} \end{aligned} \rightarrow \begin{aligned} u_1' &= \lambda_1 u_1 \\ u_2' &= \lambda_2 u_2 \\ &\vdots \end{aligned} \rightarrow u_i = c_i e^{\lambda_i t}$$

$$\rightarrow \underline{z} = P\underline{u} = \left(\cdots \underline{v}_i \cdots \right) \cdot \begin{pmatrix} \vdots \\ c_i e^{\lambda_i t} \\ \vdots \end{pmatrix} = \sum_i \underline{v}_i \cdot c_i e^{\lambda_i t}$$

y_2



$$(y_1 - 3y_2 + 2,\ 2y_1 - 4y_2 + 2)$$

$$\parallel$$

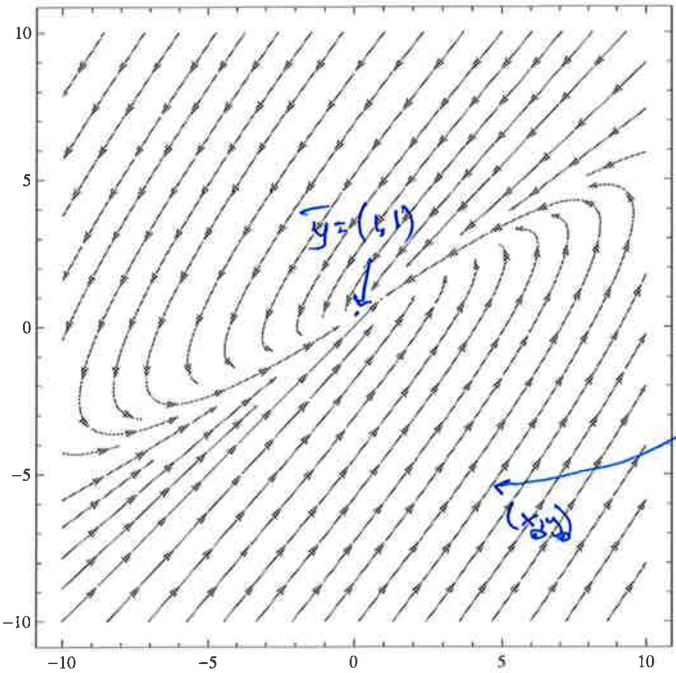
vector is (y_1', y_2')

$\rightarrow y_1$

(VectorPlot in Mathematica)

In[5]:= **StreamPlot[{x - 3 y + 2, 2 x - 4 y + 2}, {x, -10, 10}, {y, -10, 10}]** (Mathematica)

Out[5]=



From any starting point (x_0, y_0) at $t=0$, the integral curve tends to steady state $(1,1)$

Global asymptotically stable if $y \to \bar{y}$ when $t \to \infty$ for all initial states y_0.

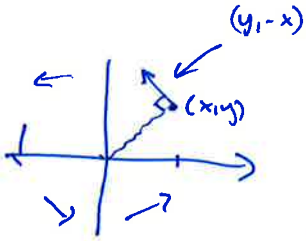
$$\Updownarrow$$

$$\lambda_1 < 0, \lambda_2 < 0, \dots, \lambda_n < 0$$

If λ_i are complex eigenvalues, the condition becomes: the real part of λ_i is negative for all i

In the case $n=2$:

$$\lambda_1, \lambda_2 < 0 \iff \begin{cases} \overset{\lambda_1+\lambda_2}{\overset{\shortparallel}{\mathrm{tr}\,A}} < 0 \\ \underset{\lambda_1 \cdot \lambda_2}{\underset{\shortparallel}{\det A}} > 0 \end{cases}$$



This characterization also holds for complex eigenvalues

Ex: $y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\lambda^2 + 1 = 0$

$\lambda^2 = \pm\sqrt{-1} = \pm i$

$\lambda_1 = \underline{i}$, $\lambda_2 = \underline{-i}$

(complex eigenvalues)

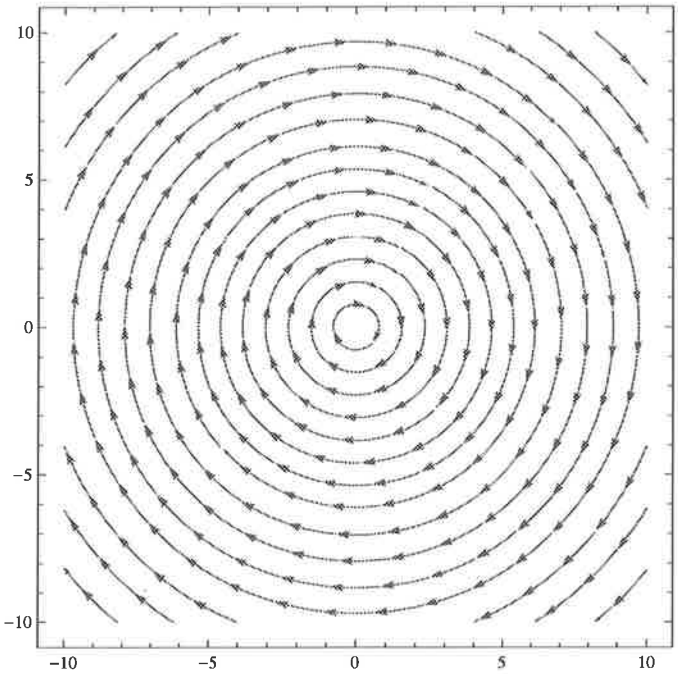
Complex numbers: $z = \underset{\text{real part}}{a} + i\underset{\text{imaginary part}}{b}$, where $a, b \in \mathbb{R}$, "$i = \sqrt{-1}$" (ie $i^2 = -1$)

$z^2 - 2z + 5 = 0$:

$$z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2}$$

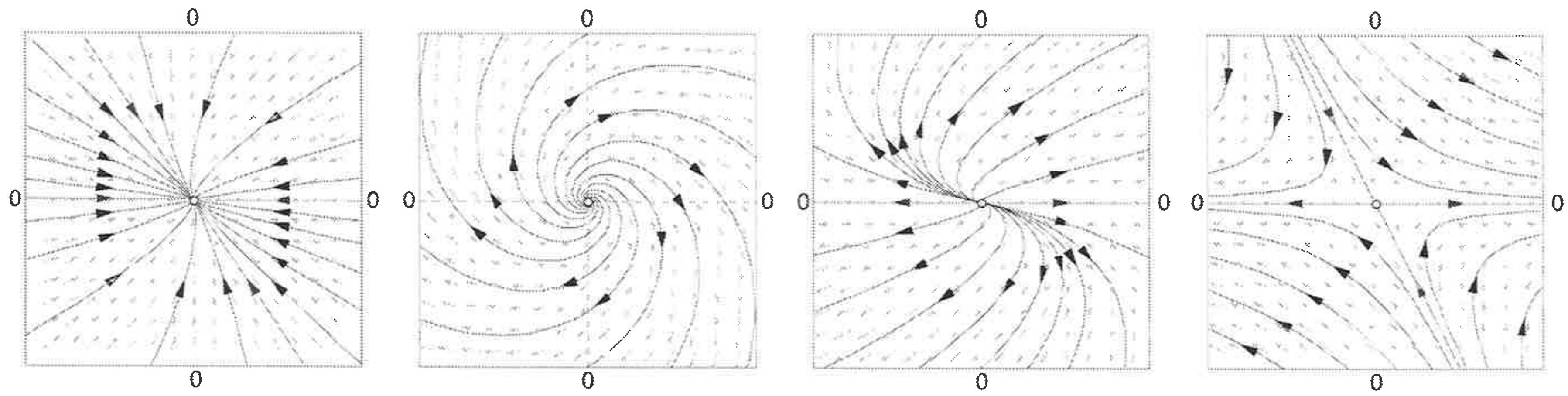
$$= 1 \pm \frac{\sqrt{16} \cdot i}{2} = 1 \pm 2i$$

$z_1 = \underline{1+2i}$, $z_2 = \underline{1-2i}$



$\underline{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{y}$ with pure imaginary eigenvalues
(real part is zero)

$\lambda = \pm i$



Some other examples of vector fields.

③ Linear approximations

$$\left.\begin{aligned} y_1' &= F(y_1, y_2) \\ y_2' &= G(y_1, y_2) \end{aligned}\right\}$$ where F, G are general (non-linear) functions

Steady state: $\underline{y} = \underline{\bar{y}}$ s.t. $F(\bar{y}) = G(\bar{y}) = 0$.

Linearization:

$$y_1' = F'_{y_1}(\bar{y}) \cdot (y_1 - \bar{y}_1) + F'_{y_2}(\bar{y}) \cdot (y_2 - \bar{y}_2)$$
$$y_2' = G'_{y_1}(\bar{y})(y_1 - \bar{y}_1) + G'_{y_2}(\bar{y}) \cdot (y_2 - \bar{y}_2)$$

$\underline{y}' = A \cdot (\underline{y} - \underline{\bar{y}})$ or $\boxed{\underline{z}' = A \cdot \underline{z}}$ with $\underline{z} = \underline{y} - \underline{\bar{y}}$ and

$$A = \begin{pmatrix} F'_{y_1} & F'_{y_2} \\ G'_{y_1} & G'_{y_2} \end{pmatrix}$$

Ex: $x' = x - 3y + 2x^2 + y^2 - xy$
$y' = 2x - y - e^{x+y} + 1$

$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is one steady state (there may be others)

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Linearization:

$$\underline{z}' = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \underline{z}$$

$\left.\begin{aligned} \det A &= -2 + 3 = 1 > 0 \\ \operatorname{tr} A &= 1 + (-2) = -1 < 0 \end{aligned}\right\}$ Globally asymptotically stable at $(0,0)$.