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Chapter 10
Computing Noncommutative Global Deformations of D-Modules

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Abstract Let $(X,\mathscr{D})$ be a D-scheme in the sense of Beilinson and Bernstein, given by an algebraic variety X and a morphism $\mathscr{O}_X \to \mathscr{D}$ of sheaves of rings on X. We consider noncommutative deformations of quasi-coherent sheaves of left $\mathscr{D}$-modules on X, and show how to compute their pro-representing hulls. As an application, we compute the noncommutative deformations of the left $\mathscr{D}_X$-module $\mathscr{O}_X$ when X is any elliptic curve.

10.1 Introduction

Let k be an algebraically closed field of characteristic 0, and let X be an algebraic variety over k, i.e. an integral separated scheme of finite type over k. A *D-algebra* in the sense of Beilinson and Bernstein [2] is a sheaf $\mathscr{D}$ of associative rings on X, together with a morphism $i : \mathscr{O}_X \to \mathscr{D}$ of sheaves of rings on X, such that the following conditions hold: (1) $\mathscr{D}$ is quasi-coherent as a left and right $\mathscr{O}_X$-module via i, and (2) for any open subset $U \subseteq X$ and any section $P \in \mathscr{D}(U)$, there is an integer $n \geq 0$ such that

$$[\dots[[P,a_1],a_2],\dots,a_n] = 0$$

for all sections $a_1,\dots,a_n \in \mathscr{O}_X(U)$, where $[P,Q] = PQ - QP$ is the commutator in $\mathscr{D}(U)$. When $\mathscr{D}$ is a D-algebra on X, the ringed space $(X,\mathscr{D})$ is called a *D-scheme*.

Let us denote the sheaf of k-linear on X by $\mathscr{D}_X$, and for any Lie algebroid $\mathfrak{g}$ of X/k, let us denote the enveloping D-algebra of $\mathfrak{g}$ by $\mathrm{U}(\mathfrak{g})$. We see that $\mathscr{D}_X$ and $\mathrm{U}(\mathfrak{g})$ are examples of noncommutative D-algebras on X, and that $\mathscr{O}_X$ is an example of a commutative D-algebra on X.

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Let us define a *D-module* on a D-scheme $(X, \mathscr{D})$ to be a quasi-coherent sheaf of left $\mathscr{D}$-modules on X. In Eriksen [3], we developed a noncommutative global deformation theory of D-modules that generalizes the usual (commutative) global deformation theory of D-modules, and the noncommutative deformation theory of modules in the affine case, due to Laudal. In Sects. 10.2–10.3, we review the essential parts of this theory, including the global Hochschild cohomology and the global obstruction calculus, all the time with a view towards concrete computations.

The purpose of this paper is to show how to apply the theory in order to compute noncommutative global deformations of interesting D-modules. In Sect. 10.4, we consider the noncommutative deformation functor

$$\mathsf{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$$

of $\mathscr{O}_X$ considered as a left $\mathscr{D}_X$-module when X is any over k. Recall that in this case, a quasi-coherent $\mathscr{D}_X$-module structure on $\mathscr{O}_X$ is the same as an integrable connection on $\mathscr{O}_X$, and according to a theorem due to André Weil, see Weil [7] and also Atiyah [1], a line bundle admits an integrable connection if and only if it has degree zero.

We show that the noncommutative deformation functor $\mathsf{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$ has pro-representing hull $H = k\langle\langle t_1, t_2 \rangle\rangle/(t_1t_2 - t_2t_1) \cong k[[t_1, t_2]]$ that is commutative, smooth and of dimension two. We also compute the corresponding versal family in concrete terms, and remark that it does not admit an algebraization.

10.2 Noncommutative Global Deformations of D-Modules

Let $(X, \mathscr{D})$ be a D-scheme, and let $\mathsf{QCoh}(\mathscr{D})$ be the category of quasi-coherent sheaves of left $\mathscr{D}$-modules on X. This is the full subcategory of $\mathsf{Sh}(X, \mathscr{D})$, the category of sheaves of left $\mathscr{D}$-modules on X, consisting of quasi-coherent sheaves. We recall that a sheaf $\mathscr{F}$ of left $\mathscr{D}$-modules on X is quasi-coherent if for every point $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x, free sheaves $\mathscr{L}_0, \mathscr{L}_1$ of left $\mathscr{D}|_U$-modules on U, and an exact sequence $0 \leftarrow \mathscr{F}|_U \leftarrow \mathscr{L}_0 \leftarrow \mathscr{L}_1$ of sheaves of left $\mathscr{D}|_U$-modules on U. We shall refer to the quasi-coherent sheaves of left $\mathscr{D}$-modules on X as *D-modules* on the D-scheme $(X, \mathscr{D})$.

For any D-scheme $(X, \mathscr{D})$, $\mathsf{QCoh}(\mathscr{D})$ is an Abelian k-category, and we consider noncommutative deformations in $\mathsf{QCoh}(\mathscr{D})$. For a finite family $\mathscr{F} = \{\mathscr{F}_1, \dots, \mathscr{F}_p\}$ of quasi-coherent left $\mathscr{D}$-modules on X, there is a noncommutative deformation functor $\mathsf{Def}^{qc}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$ of $\mathscr{F}$ in $\mathsf{QCoh}(\mathscr{D})$, generalizing the noncommutative deformation functor of modules introduced in Laudal [5]. We shall provide a brief description of $\mathsf{Def}^{qh}_{\mathscr{F}}$ below; see Eriksen [3] for further details.

We recall that the objects of the category a_p of p-pointed *noncommutative Artin rings* are Artinian rings R, together with pairs of structural morphisms $f : k^p \to R$ and $g : R \to k^p$, such that $g \circ f = \mathrm{id}$ and the radical $I(R) = \mathrm{Ker}(g)$ is nilpotent. The morphisms are the natural commutative diagrams. For any $R \in \mathsf{a}_p$, there are

p isomorphism classes of simple left R-modules, represented by $\{k_1, k_2, \dots, k_p\}$, where $k_i = 0 \times \cdots \times k \times \cdots \times 0$ is the i'th projection of k^p for $1 \leq i \leq p$.

We remark that any $R \in \mathsf{a}_p$ is a $p \times p$ *matrix ring*, in the sense that there are p indecomposable idempotents $\{e_1, \dots, e_p\}$ in R with $e_1 + \cdots + e_p = 1$ and a decomposition $R = \oplus R_{ij}$, given by $R_{ij} = e_i R e_j$, such that elements of R multiply as matrices. We shall therefore use matrix notation, and write $R = (R_{ij})$ when $R \in \mathsf{a}_p$, and $(V_{ij}) = \oplus V_{ij}$ when $\{V_{ij} : 1 \leq i, j \leq p\}$ is any family of vector spaces.

For any $R \in \mathsf{a}_p$, a lifting of $\mathscr{F}$ to R is a quasi-coherent sheaf $\mathscr{F}_R$ of left $\mathscr{D}$-modules with a compatible right R-module structure, together with isomorphisms $\eta_i : \mathscr{F}_R \otimes_R k_i \to \mathscr{F}_i$ in $\mathsf{QCoh}(\mathscr{D})$ for $1 \leq i \leq p$, such that $\mathscr{F}_R(U) \cong (\mathscr{F}_i(U) \otimes_k R_{ij})$ as right R-modules for all open subsets $U \subseteq X$. We say that two liftings $(\mathscr{F}_R, \eta_i)$ and $(\mathscr{F}'_R, \eta'_i)$ are equivalent if there is an isomorphism $\tau : \mathscr{F}_R \to \mathscr{F}'_R$ of $\mathscr{D}$-R bimodules on X such that $\eta'_i \circ (\tau \otimes_k k_i) = \eta_i$ for $1 \leq i \leq p$, and denote the set of equivalence classes of liftings of $\mathscr{F}$ to R by $\mathrm{Def}^{qc}_{\mathscr{F}}(R)$. This defines the noncommutative deformation functor $\mathrm{Def}^{qc}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$.

10.3 Computing Noncommutative Global Deformations

Let $(X, \mathscr{D})$ be a D-scheme, and let U be an open affine cover of X that is finite and closed under intersections. We shall explain how to compute noncommutative deformations in $\mathsf{QCoh}(\mathscr{D})$ effectively using the open cover U.

We may consider U as a small category, where the objects are the open subsets $U \in \mathsf{U}$, and the morphisms from U to V are the (opposite) inclusions $U \supseteq V$. There is a natural forgetful functor $\mathsf{QCoh}(\mathscr{D}) \to \mathsf{PreSh}(\mathsf{U}, \mathscr{D})$, where $\mathsf{PreSh}(\mathsf{U}, \mathscr{D})$ is the Abelian k-category of (covariant) presheaves of left $\mathscr{D}$-modules on U. For any finite family $\mathscr{F}$ in $\mathsf{QCoh}(\mathscr{D})$, this forgetful functor induces an isomorphism of noncommutative deformation functors $\mathrm{Def}^{qc}_{\mathscr{F}} \to \mathrm{Def}_{\mathscr{F}}$, where $\mathrm{Def}^{qc}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$ is the noncommutative deformation functor of $\mathscr{F}$ in $\mathsf{QCoh}(\mathscr{D})$ defined in Sect. 10.2, and $\mathrm{Def}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$ is the noncommutative deformation functor of $\mathscr{F}$ in $\mathsf{PreSh}(\mathsf{U}, \mathscr{D})$, defined in a similar way; see Eriksen [3] for details.

Theorem 10.1. *Let $(X, \mathscr{D})$ be a D-scheme, and let $\mathscr{F} = \{\mathscr{F}_1, \dots, \mathscr{F}_p\}$ be a finite family in $\mathsf{QCoh}(\mathscr{D})$. If the global Hochschild cohomology*

$$(\mathrm{HH}^n(\mathsf{U}, \mathscr{D}, \mathrm{Hom}_k(\mathscr{F}_j, \mathscr{F}_i)))$$

has finite k-dimension for $n = 1, 2$, then the noncommutative deformation functor $\mathrm{Def}^{qc}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$ has a pro-representing hull $H = H(\mathscr{F})$, completely determined by $(\mathrm{HH}^n(\mathsf{U}, \mathscr{D}, \mathrm{Hom}_k(\mathscr{F}_j, \mathscr{F}_i)))$ for $n = 1, 2$ and their generalized Massey products.

In fact, there is a constructive proof of the fact that $\mathrm{Def}_{\mathscr{F}} : \mathsf{a}_p \to \mathsf{Sets}$ of $\mathscr{F}$ in $\mathsf{PreSh}(\mathsf{U}, \mathscr{D})$ has a pro-representing hull; see Eriksen [3] for details. The construction uses the global Hochschild cohomology $(\mathrm{HH}^n(\mathsf{U}, \mathscr{D}, \mathrm{Hom}_k(\mathscr{F}_j, \mathscr{F}_i)))$ for $n = 1, 2$, and the obstruction calculus of $\mathrm{Def}_{\mathscr{F}}$, which can be expressed in terms of generalized Massey products on these cohomology groups. We give a brief description of the global Hochschild cohomology and the obstruction calculus below.

10.3.1 Cohomology

For any presheaves $\mathscr{F},\mathscr{G}$ of left $\mathscr{D}$-modules on U, we recall the definition of *global Hochschild cohomology* $\mathrm{HH}^n(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G}))$ of $\mathscr{D}$ with values in the bimodule $\mathrm{Hom}_k(\mathscr{F},\mathscr{G})$ on U. For any inclusion $U \supseteq V$ in U, we consider the Hochschild complex $\mathsf{HC}^*(\mathscr{D}(U),\mathrm{Hom}_k(\mathscr{F}(U),\mathscr{G}(V))$ of $\mathscr{D}(U)$ with values in the bimodule $\mathrm{Hom}_k(\mathscr{F}(U),\mathscr{G}(V))$. We define the category $\mathsf{Mor}\,\mathsf{U}$ to have (opposite) inclusions $U \supseteq V$ in U as its objects, and nested inclusions $U' \supseteq U \supseteq V \supseteq V'$ in U as its morphisms from $U \supseteq V$ to $U' \supseteq V'$. It follows that we may consider the Hochschild complex

$$\mathsf{HC}^*(\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G})) : \mathsf{Mor}\,\mathsf{U} \to \mathsf{Compl}(k)$$

as a functor on $\mathsf{Mor}\,\mathsf{U}$. The global Hochschild complex is the total complex of the double complex $\mathsf{D}^{**} = \mathsf{D}^*(\mathsf{U},\mathsf{HC}^*(\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G})))$, where

$$\mathsf{D}^*(\mathsf{U},-) : \mathsf{PreSh}(\mathsf{Mor}\,\mathsf{U},k) \to \mathsf{Compl}(k)$$

is the resolving complex of the projective limit functor; see Laudal [4] for details. We denote the global Hochschild complex by $\mathsf{HC}^*(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G}))$, and define the global Hochschild cohomology $\mathrm{HH}^n(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G}))$ to be its cohomology.

Note that $\mathsf{H}^n(\mathsf{HC}^*(\mathscr{D}(U),\mathrm{Hom}_k(\mathscr{F}(U),\mathscr{G}(V)))) \cong \mathrm{Ext}^n_{\mathscr{D}(U)}(\mathscr{F}(U),\mathscr{G}(V))$ for any $U \supseteq V$ in U since k is a field. Hence there is a spectral sequence converging to the global Hochschild cohomology $\mathrm{HH}^n(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F},\mathscr{G}))$ with

$$E_2^{pq} = \mathsf{H}^p(\mathsf{U},\mathrm{Ext}^q_{\mathscr{D}}(\mathscr{F},\mathscr{G})), \tag{10.1}$$

where $\mathsf{H}^p(\mathsf{U},-) = \mathsf{H}^p(\mathsf{D}^*(\mathsf{U},-))$ and $\mathrm{Ext}^q_{\mathscr{D}}(\mathscr{F},\mathscr{G}) : \mathsf{Mor}\,\mathsf{U} \to \mathsf{Mod}(k)$ is the functor on $\mathsf{Mor}\,\mathsf{U}$ given by $\{U \supseteq V\} \mapsto \mathrm{Ext}^q_{\mathscr{D}(U)}(\mathscr{F}(U),\mathscr{G}(V))$ for all $q \geq 0$.

10.3.2 Obstruction Calculus

Let $R \in \mathsf{a}_p$ and let $I = I(R)$ be the radical of R. For any lifting $\mathscr{F}_R \in \mathrm{Def}_{\mathscr{F}}(R)$ of the family $\mathscr{F}$ in $\mathsf{PreSh}(\mathsf{U},\mathscr{D})$ to R, we have that $\mathscr{F}_R(U) \cong (\mathscr{F}_i(U) \otimes_k R_{ij})$ as a right R-module for all $U \in \mathsf{U}$. Moreover, the lifting $\mathscr{F}_R$ is completely determined by the left multiplication of $\mathscr{D}(U)$ on $\mathscr{F}_R(U)$ for all $U \in \mathsf{U}$ and the restriction map $\mathscr{F}_R(U) \to \mathscr{F}_R(V)$ for all $U \supseteq V$ in U. Let us write $Q^R(U,V) = (\mathrm{Hom}_k(\mathscr{F}_j(U),\mathscr{F}_i(V) \otimes_k R_{ij}))$ and $Q^R(U) = Q^R(U,U)$ for all $U \supseteq V$ in U. Then $\mathscr{F}_R \in \mathrm{Def}_{\mathscr{F}}(R)$ is completely described by the following data:

1. For all $U \in \mathsf{U}$, a k-algebra homomorphism $L(U) : \mathscr{D}(U) \to Q^R(U)$ satisfying $L(U)(P)(f_j) = Pf_j \otimes e_j + (\mathscr{F}_i(U) \otimes_k I_{ij})$ for all $P \in \mathscr{D}(U)$, $f_j \in \mathscr{F}_j(U)$.
2. For all inclusions $U \supseteq V$ in U, a restriction map $L(U,V) \in Q^R(U,V)$ satisfying $L(U,V)(f_j) = (f_j|_V) \otimes e_j + (\mathscr{F}_i(V) \otimes I_{ij})$ for all $f_j \in \mathscr{F}_j(U)$ and $L(U,V) \circ L(U)(P) = L(V)(P|_V) \circ L(U,V)$ for all $P \in \mathscr{D}(U)$.
3. For all inclusions $U \supseteq V \supseteq W$ in U, $L(V,W)L(U,V) = L(U,W)$ and $L(U,U) = \mathrm{id}$.

A *small surjection* in a_p is a surjective morphism $u : R \to S$ in a_p such that $IK = KI = 0$, where $K = \mathrm{Ker}(u)$ and $I = I(R)$ is the radical of R. To describe the obstruction calculus of $\mathrm{Def}_{\mathscr{F}}$, it is enough to consider the following problem: Given a small surjection $u : R \to S$ and a deformation $\mathscr{F}_S \in \mathrm{Def}_{\mathscr{F}}(S)$, what are the possible liftings of $\mathscr{F}_S$ to R? The answer is given by the following proposition; see Eriksen [3] for details:

Proposition 10.1. *Let $u : R \to S$ be a small surjection in a_p with kernel K, and let $\mathscr{F}_S \in \mathrm{Def}_{\mathscr{F}}(S)$ be a deformation. Then there exists a canonical obstruction*

$$o(u,\mathscr{F}_S) \in (\mathrm{HH}^2(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F}_j,\mathscr{F}_i)) \otimes_k K_{ij})$$

such that $o(u,\mathscr{F}_S) = 0$ if and only if there is a deformation $\mathscr{F}_R \in \mathrm{Def}_{\mathscr{F}}(R)$ lifting $\mathscr{F}_S$ to R. Moreover, if $o(u,\mathscr{F}_S) = 0$, then there is a transitive and effective action of $(\mathrm{HH}^1(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{F}_j,\mathscr{F}_i)) \otimes_k K_{ij})$ on the set of liftings of $\mathscr{F}_S$ to R.

In fact, let $\mathscr{F}_S \in \mathrm{Def}_{\mathscr{F}}(S)$ be given by $L^S(U) : \mathscr{D}(U) \to Q^S(U)$ and $L^S(U,V) \in Q^S(U,V)$ for all $U \supseteq V$ in U, and let $\sigma : S \to R$ be a k-linear section of $u : R \to S$ such that $\sigma(e_i) = e_i$ and $\sigma(S_{ij}) \subseteq R_{ij}$ for $1 \le i,j \le p$. We consider $L^R(U) : \mathscr{D}(U) \to Q^R(U)$ given by $L^R(U) = \sigma \circ L^S(U)$ and $L^R(U,V) = \sigma(L^S(U,V))$ for all $U \supseteq V$ in U. The obstruction $o(U,\mathscr{F}_S)$ for lifting $\mathscr{F}_S$ to R is given by

1. $(P,Q) \mapsto L^R(U)(PQ) - L^R(U)(P) \circ L^R(U)(Q)$ for all $U \in \mathsf{U}$ and $P,Q \in \mathscr{D}(U)$
2. $P \mapsto L^R(U,V) \circ L^R(U)(P) - L^R(V)(P|_V) \circ L^R(U,V)$ for all $U \supseteq V$ in U, $P \in \mathscr{D}(U)$
3. $L^R(V,W) \circ L^R(U,V) - L^R(U,W)$ for all $U \supseteq V \supseteq W$ in U

We see that these expressions are exactly the obstructions for $L^R(U)$ and $L^R(U,V)$ to satisfy conditions (1)–(3) in the characterization of $\mathrm{Def}_{\mathscr{F}}(R)$ given above.

10.4 Calculations for D-Modules on Elliptic Curves

Let $X \subseteq \mathbf{P}^2$ be the irreducible projective plane curve given by the homogeneous equation $f = 0$, where $f = y^2z - x^3 - axz^2 - bz^3$ for fixed parameters $(a,b) \in k^2$. We assume that $\Delta = 4a^3 + 27b^2 \neq 0$, so that X is smooth and therefore an elliptic curve over k. We shall compute the noncommutative deformations of $\mathscr{O}_X$, considered as a quasi-coherent left $\mathscr{D}_X$-module via the natural left action of $\mathscr{D}_X$ on $\mathscr{O}_X$.

We choose an open affine cover $\mathsf{U} = \{U_1, U_2, U_3\}$ of X closed under intersections, given by $U_1 = D_+(y)$, $U_2 = D_+(z)$ and $U_3 = U_1 \cap U_2$. We recall that the open subset $D_+(h) \subseteq X$ is given by $D_+(h) = \{p \in X : h(p) \neq 0\}$ for $h = y$ or $h = z$. It follows from the results in Sect. 10.3 that the noncommutative deformation functor $\mathrm{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$ has a pro-representing hull H, completely determined by the global Hochschild cohomology groups and some generalized Massey products on them. We shall therefore compute $\mathrm{HH}^n(\mathsf{U}, \mathscr{D}, \mathrm{End}_k(\mathscr{O}_X))$ for $n = 1, 2$.

It is known that $\mathscr{D}_X(U)$ is a simple Noetherian ring of global dimension one and that $\mathscr{O}_X(U)$ is a simple left $\mathscr{D}_X(U)$-module for any open affine subset $U \subseteq X$; see for instance Smith and Stafford [6]. Hence, the functor $\mathrm{Ext}^q_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X) : \mathsf{Mor}\,\mathsf{U} \to \mathsf{Mod}(k)$ satisfies $\mathrm{Ext}^q_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X) = 0$ for $q \geq 2$ and $\mathrm{End}_{\mathscr{D}_X}(\mathscr{O}_X) = k$. Since the spectral sequence for global Hochschild cohomology given in Sect. 10.3 degenerates,

$$\mathrm{HH}^n(\mathsf{U}, \mathscr{D}_X, \mathrm{End}_k(\mathscr{O}_X)) \cong \mathsf{H}^{n-1}(\mathsf{U}, \mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X)) \text{ for } n \geq 1$$
$$\mathrm{HH}^0(\mathsf{U}, \mathscr{D}_X, \mathrm{End}_k(\mathscr{O}_X)) \cong k$$

We compute $\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X)$ and use the result to find $\mathsf{H}^{n-1}(\mathsf{U}, \mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X))$ for $n = 1, 2$.

Let $A_i = \mathscr{O}_X(U_i)$ and $D_i = \mathscr{D}_X(U_i)$ for $i = 1, 2, 3$. We see that $A_1 \cong k[x,z]/(f_1)$ and $A_2 \cong k[x,y]/(f_2)$, where $f_1 = z - x^3 - axz^2 - bz^3$ and $f_2 = y^2 - x^3 - ax - b$. Moreover, we have that $\mathrm{Der}_k(A_i) = A_i \partial_i$ and $D_i = A_i \langle \partial_i \rangle$ for $i = 1, 2$, where

$$\partial_1 = (1 - 2axz - 3bz^2)\, \partial/\partial x + (3x^2 + az^2)\, \partial/\partial z$$
$$\partial_2 = -2y\, \partial/\partial x - (3x^2 + a)\, \partial/\partial y$$

We choose an isomorphism $A_3 \cong k[x,y,y^{-1}]/(f_3)$ with $f_3 = f_2$ on the intersection $U_3 = U_1 \cap U_2$, and see that $\mathrm{Der}_k(A_3) = A_3 \partial_3$ and $D_3 = A_3\langle \partial_3 \rangle$ for $\partial_3 = \partial_2$. The restriction maps of $\mathscr{O}_X$ and $\mathscr{D}_X$, considered as presheaves on U, are given by

$$x \mapsto xy^{-1},\ z \mapsto y^{-1},\ \partial_1 \mapsto \partial_3$$

for the inclusion $U_1 \supseteq U_3$, and the natural localization map for $U_2 \supseteq U_3$. Finally, we find a free resolution of A_i as a left D_i-module for $i = 1, 2, 3$, given by

$$0 \leftarrow A_i \leftarrow D_i \xleftarrow{\cdot \partial_i} D_i \leftarrow 0$$

and use this to compute $\mathrm{Ext}^1_{D_i}(A_i, A_j) \cong \mathrm{coker}(\partial_i|_{U_j} : A_j \to A_j)$ for $U_i \supseteq U_j$ in U. We see that $\mathrm{Ext}^1_{D_i}(A_i, A_3) \cong \mathrm{coker}(\partial_3 : A_3 \to A_3)$ is independent of i, and find the following k-linear bases for $\mathrm{Ext}^1_{D_i}(A_i, A_j)$:

	$a \neq 0$:	$a = 0$:
$U_1 \supseteq U_1$	$1, z, z^2, z^3$	$1, z, x, xz$
$U_2 \supseteq U_2$	$1, y^2$	$1, x$
$U_3 \supseteq U_3$	$x^2y^{-1}, 1, y^{-1}, y^{-2}, y^{-3}$	$x^2y^{-1}, 1, y^{-1}, x, xy^{-1}$

The functor $\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X) : \mathsf{Mor}\ \mathsf{U} \to \mathsf{Mod}(k)$ defines the diagram in $\mathsf{Mod}(k)$ below, where the maps are induced by the restriction maps on $\mathscr{O}_X$:

$$\begin{tikzcd}
\mathrm{Ext}^1_{D_1}(A_1,A_1) \arrow[d] & & \mathrm{Ext}^1_{D_2}(A_2,A_2) \arrow[d] \\
\mathrm{Ext}^1_{D_1}(A_1,A_3) \arrow[r, equal] & \mathrm{Ext}^1_{D_3}(A_3,A_3) \arrow[r, equal] & \mathrm{Ext}^1_{D_2}(A_2,A_3)
\end{tikzcd}$$

We use that $15\,y^2 = \Delta\, y^{-2}$ in $\mathrm{Ext}^1_{D_3}(A_3,A_3)$ when $a \neq 0$ and that $-3b\,xy^{-2} = x$ in $\mathrm{Ext}^1_{D_3}(A_3,A_3)$ when $a = 0$ to describe these maps in the given bases. We compute $\mathsf{H}^{n-1}(\mathsf{U},\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X))$ for $n = 1,2$ using the resolving complex $\mathsf{D}^*(\mathsf{U},-)$; see Laudal [4] for definitions. In particular, we identify $\mathsf{H}^0(\mathsf{U},\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X))$ with the set of all pairs (h_1,h_2) with $h_i \in \mathrm{Ext}^1_{D_i}(A_i,A_i)$ that satisfies $h_1|_{U_3} = h_2|_{U_3}$, and we identify $\mathsf{H}^1(\mathsf{U},\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X))$ with the set of all pairs (h_{13},h_{23}) with $h_{ij} \in \mathrm{Ext}^1_{D_i}(A_i,A_j)$, modulo the pairs of the form $(h_1|_{U_3} - h_3, h_2|_{U_3} - h_3)$ for triples (h_1,h_2,h_3) with $h_i \in \mathrm{Ext}^1_{D_i}(A_i,A_i)$. We find the following k-linear bases:

	$a \neq 0:$	$a = 0:$
$n=1$	$\xi_1 = (1,1,1)$, $\xi_2 = (\Delta z^2, 15y^2, \Delta y^{-2})$	$\xi_1 = (1,1,1)$, $\xi_2 = (-3b\,xz, x, x)$
$n=2$	$\omega = (0,0,0,0,6ax^2y^{-1})$	$\omega = (0,0,0,0,x^2y^{-1})$

We recall that ξ_1,ξ_2 and ω are represented by cocycles of degree $p = 0$ and $p = 1$ in the resolving complex $\mathsf{D}^*(\mathsf{U},\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X))$, where

$$\mathsf{D}^p(\mathsf{U},\mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X)) = \prod_{U_0 \supseteq \cdots \supseteq U_p} \mathrm{Ext}^1_{\mathscr{D}_X}(\mathscr{O}_X,\mathscr{O}_X)(U_0 \supseteq U_p)$$

and the product is indexed by the set $\{U_1 \supseteq U_1, U_2 \supseteq U_2, U_3 \supseteq U_3\}$ when $p = 0$, and $\{U_1 \supseteq U_1, U_2 \supseteq U_2, U_3 \supseteq U_3, U_1 \supseteq U_3, U_2 \supseteq U_3\}$ when $p = 1$.

This shows that for any elliptic curve X over k, the noncommutative deformation functor $\mathrm{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$ of the left $\mathscr{D}_X$-module $\mathscr{O}_X$ has a two-dimensional tangent space (since $\mathrm{HH}^1(\mathsf{U},\mathscr{D}_X,\mathrm{End}_k(\mathscr{O}_X)) \cong k^2$), and a one-dimensional obstruction space (since $\mathrm{HH}^2(\mathsf{U},\mathscr{D}_X,\mathrm{End}_k(\mathscr{O}_X)) \cong k$), and a pro-representing hull $H = k\langle\langle t_1,t_2\rangle\rangle/(F)$ for some noncommutative power series $F \in k\langle\langle t_1,t_2\rangle\rangle$.

We shall compute the noncommutative power series F and the versal family $\mathscr{F}_H \in \mathrm{Def}_{\mathscr{O}_X}(H)$ using the obstruction calculus for $\mathrm{Def}_{\mathscr{O}_X}$. We choose base vectors t_1^*, t_2^* in $\mathrm{HH}^1(\mathsf{U},\mathscr{D},\mathrm{Hom}_k(\mathscr{O}_X,\mathscr{O}_X))$, and representatives $(\psi_l,\tau_l) \in \mathsf{D}^{01} \oplus \mathsf{D}^{10}$ of t_l^* for $l = 1,2$, where $\mathsf{D}^{pq} = \mathsf{D}^p(\mathsf{U},\mathsf{HC}^q(\mathscr{D},\mathrm{End}_k(\mathscr{O}_X)))$. We may choose $\psi_l(U_i)$ to be the derivation defined by

$$\psi_l(U_i)(P_i) = \begin{cases} 0 & \text{if } P_i \in A_i \\ \xi_l(U_i) \cdot \mathrm{id}_{A_i} & \text{if } P_i = \partial_i \end{cases}$$

for $l = 1,2$ and $i = 1,2,3$, and $\tau_l(U_i \supseteq U_j)$ to be the multiplication operator in $\mathrm{Hom}_{A_i}(A_i,A_j) \cong A_j$ given by $\tau_1 = 0$, $\tau_2(U_i \supseteq U_i) = 0$ for $i = 1,2,3$ and

$a \neq 0$:	$a = 0$:
$\tau_2(U_1 \supseteq U_3) = 0$ $\tau_2(U_2 \supseteq U_3) = -4a^2y^{-1} - 3xy + 9bxy^{-1} - 6ax^2y^{-1}$	$\tau_2(U_1 \supseteq U_3) = x^2y^{-1}$ $\tau_2(U_2 \supseteq U_3) = 0$

Let $\mathsf{a}_1(n)$ be the full subcategory of a_1 consisting of all R such that $I(R)^n = 0$ for $n \geq 2$. The restriction of $\mathsf{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$ to $\mathsf{a}_1(2)$ is represented by $(H_2, \mathscr{F}_{H_2})$, where $H_2 = k\langle t_1,t_2\rangle/(t_1,t_2)^2$ and the deformation $\mathscr{F}_{H_2} \in \mathsf{Def}_{\mathscr{O}_X}(H_2)$ is defined by $\mathscr{F}_{H_2}(U_i) = A_i \otimes_k H_2$ as a right H_2-module for $i = 1,2,3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

for $i = 1,2,3$ and for all $P_i \in D_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j)\, m_i|_{U_j} \otimes t_2$$

for $i = 1,2$, $j = 3$ and for all $m_i \in A_i$.

We try to lift the family $\mathscr{F}_{H_2} \in \mathsf{Def}_{\mathscr{O}_X}(H_2)$ to $R = k\langle\langle t_1,t_2\rangle\rangle/(t_1,t_2)^3$. We let $\mathscr{F}_R(U_i) = A_i \otimes_k R$ as a right R-module for $i = 1,2,3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

for $i = 1,2,3$ and for all $P_i \in D_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j)\, m_i|_{U_j} \otimes t_2 + \frac{\tau_2(U_i \supseteq U_j)^2}{2}\, m_i|_{U_j} \otimes t_2^2$$

for $i = 1,2$, $j = 3$ and for all $m_i \in A_i$. We see that $\mathscr{F}_R(U_i)$ is a left $\mathscr{D}_X(U_i)$-module for $i = 1,2,3$, and that $t_1t_2 - t_2t_1 = 0$ is a necessary and sufficient condition for $\mathscr{D}_X$-linearity of the restriction maps for the inclusions $U_1 \supseteq U_3$ and $U_2 \supseteq U_3$. This implies that $\mathscr{F}_R$ is not a lifting of $\mathscr{F}_{H_2}$ to R. But if we define the quotient $H_3 = R/(t_1t_2 - t_2t_1)$, we see that the family $\mathscr{F}_{H_3} \in \mathsf{Def}_{\mathscr{O}_X}(H_3)$ induced by $\mathscr{F}_R$ is a lifting of $\mathscr{F}_{H_2}$ to H_3.

In fact, we claim that the restriction of $\mathsf{Def}_{\mathscr{O}_X} : \mathsf{a}_1 \to \mathsf{Sets}$ to $\mathsf{a}_1(3)$ is represented by $(H_3, \mathscr{F}_{H_3})$. One way to prove this is to show that it is not possible to find any lifting $\mathscr{F}'_R \in \mathsf{Def}_{\mathscr{O}_X}(R)$ of $\mathscr{F}_{H_2}$ to R. Another approach is to calculate the cup products $< t_i^*, t_j^* >$ in global Hochschild cohomology for $i,j = 1,2$, and this gives

$$< t_1^*, t_2^* >= o^*, \; < t_2^*, t_1^* >= -o^* \qquad \text{for } a \neq 0$$
$$< t_1^*, t_2^* >= o^*, \; < t_2^*, t_1^* >= -o^* \qquad \text{for } a = 0$$

where $o^* \in \mathrm{HH}^2(\mathsf{U}, \mathscr{D}_X, \mathrm{End}_k(\mathscr{O}_X))$ is the base vector corresponding to ω. This implies that $F = t_1t_2 - t_2t_1 + (t_1, t_2)^3$, since all other cup products vanish.

Let $H = k\langle\langle t_1, t_2\rangle\rangle/(t_1t_2 - t_2t_1)$. We shall show that it is possible to find a lifting $\mathscr{F}_H \in \mathrm{Def}_{\mathscr{O}_X}(H)$ of $\mathscr{F}_{H_3}$ to H. We let $\mathscr{F}_H(U_i) = A_i \widehat{\otimes}_k H$ as a right H-module for $i = 1,2,3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

for $i = 1,2,3$ and for all $P_i \in \mathscr{D}_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto \sum_{n=0}^{\infty} \frac{\tau_2(U_i \supseteq U_j)^n}{n!} \, m_1|_{U_j} \otimes t_2^n = \exp(\tau_2(U_i \supseteq U_j) \otimes t_2) \cdot (m_1|_{U_j} \otimes 1)$$

for $i = 1,2$, $j = 3$ and for all $m_i \in A_i$. This implies that $(H, \mathscr{F}_H)$ is the pro-representing hull of $\mathrm{Def}_{\mathscr{O}_X}$, and that $F = t_1t_2 - t_2t_1$. We remark that the versal family $\mathscr{F}_H$ does not admit an algebraization, i.e. an algebra H_{alg} of finite type over k such that H is a completion of H_{alg}, together with a deformation in $\mathrm{Def}_{\mathscr{O}_X}(H_{\mathrm{alg}})$ that induces the versal family $\mathscr{F}_H \in \mathrm{Def}_{\mathscr{O}_X}(H)$.

Finally, we mention that there is an algorithm for computing the pro-representing hull H and the versal family $\mathscr{F}_H$ using the cup products and higher generalized Massey products on global Hochschild cohomology. We shall describe this algorithm in a forthcoming paper. In many situations, it is necessary to use the full power of this machinery to compute noncommutative deformation functors.

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