Chapter 9
Connections on Modules over Singularities of Finite and Tame CM Representation Type

Eivind Eriksen and Trond Stølen Gustavsen

Abstract Let R be the local ring of a singular point of a complex analytic space, and let M be an R-module. Under what conditions on R and M is it possible to find a connection on M? To approach this question, we consider maximal Cohen–Macaulay (MCM) modules over CM algebras that are isolated singularities, and review an obstruction theory implemented in the computer algebra system Singular. We report on results, with emphasis on singularities of finite and tame CM representation type.

9.1 Introduction

Let R be the local ring of a complex analytic space, and let M be an R-module. In this paper we discuss the existence of a connection on M, i.e. an R-linear map $\nabla : \mathrm{Der}_C(R) \to \mathrm{End}_C(M)$ that satisfies the Leibniz rule. These connections are related to the topology of the singularity via the Riemann–Hilbert correspondence. In the case of normal surface singularities, this relationship is particularly strong, and it is described explicitly in Gustavsen and Ile [16]. However, it is a delicate problem to describe these connections, or even to determine when such a connection exists.

To our knowledge, Kahn was the first to consider this question, see Chap. 5 in Kahn [18]. Using known properties of vector bundles on elliptic curves and an analysis of the representations of the local fundamental group of a simple elliptic surface singularity, he was able to show that any maximal Cohen–Macaulay module

E. Eriksen
Oslo University College, P.O. Box 4, St Olavs Plass, 0130 Oslo, Norway
e-mail: eeriksen@hio.no

T. S. Gustavsen
BI Norwegian School of Management, 0442 Oslo, Norway
e-mail: trond.s.gustavsen@bi.no

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on such a singularity admits an integrable connection. In the present paper, the emphasis is on more algebraic methods which enable us to treat a more diverse class of singularities.

After some preliminary material in Sect. 2, we develop an algebraic approach to the question of existence of connections in Sect. 3. This approach is given in terms of an obstruction theory, which we have implemented as the library CONN.LIB [9] in the computer Algebra system SINGULAR 3.0 [14]. We focus on the case of maximal Cohen–Macaulay (MCM) modules over CM algebras that are isolated singularities, and discuss some results that we have obtained. In the case of normal surface singularities, we show how the MCM modules with connections are related to the topology of the singularity.

Our investigations indicate that it is interesting to consider this notion of connections on modules over singularities, but they also raise many questions, and some of these seem difficult to answer in general. Nevertheless, we are able to give some interesting results and examples. For the simple curve singularities, all MCM modules admit a connection. However, there are other curve singularities of finite CM type with MCM modules that do not admit a connection. In dimension two, all MCM modules over any CM finite singularity admit a connection. In higher dimensions, it seems that very few singularities have the property that all MCM modules admit a connection. For instance, an MCM module over a simple singularity in dimension $d \geq 3$ admits a connection if and only if it is free.

Existence of connections on modules has previously been considered by several authors, including Buchweitz, Christophersen, Ile, Kahn, Källström, Laudal and Maakestad; see for instance [8, 16, 18, 19, 22].

9.2 Preliminaries

We will denote by R a Cohen–Macaulay local $\mathbb{C}$-algebra which is an integral domain, and we will assume that $R \cong \mathscr{O}_{X,x}$ where (X,x) is an isolated singularity of an analytic space X. We can embed $(X,x) \subset (\mathbb{C}^n, 0)$ and thus $R \cong \mathbb{C}\{x_1,\dots,x_n\}/(f_1,\dots,f_m)$.

A finitely generated R-module M is maximal Cohen–Macaulay if $\operatorname{depth} M = \dim R$, and we denote by $\operatorname{Der}_{\mathbb{C}}(R) \cong \operatorname{Hom}_R(\Omega^1_R, R)$ the module of derivations on R. Note that $\operatorname{Der}_{\mathbb{C}}(R)$ is a left R-module and a $\mathbb{C}$-Lie-algebra.

9.2.1 Connections

Let M be a finitely generated R-module. An *action of* $\operatorname{Der}_{\mathbb{C}}(R)$ on M is a $\mathbb{C}$-linear map $\nabla : \operatorname{Der}_{\mathbb{C}}(R) \to \operatorname{End}_{\mathbb{C}}(M)$ which for all $a \in R$, $m \in M$ and $D \in \operatorname{Der}_{\mathbb{C}}(R)$ satisfy the *Leibniz rule*

$$\nabla_D(am) = a\nabla_D(m) + D(a)m. \tag{9.1}$$

An action $\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M)$ of $\mathrm{Der}_{\mathbb{C}}(R)$ on M is said to be a *connection* if it is R-linear. We shall see that there are cases where M admits an action of $\mathrm{Der}_{\mathbb{C}}(R)$, but not a connection.

A connection $\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M)$ is *integrable* if it is a Lie-algebra homomorphism, i.e. if $\nabla([D_1, D_2]) = [\nabla(D_1), \nabla(D_2)]$.

We recall that when R is regular, a *connection* on M in the sense of André Weil is defined as a $\mathbb{C}$-linear map $\nabla : M \to M \otimes_R \Omega^1_R$ such that $\nabla(am) = a\nabla(m) + m \otimes d(a)$ for all $a \in R$, $m \in M$, where $d : R \to \Omega^1_R$ is the universal derivation, see for instance Katz [20]. Moreover, the *curvature* of ∇ is defined as the R-linear map $K_\nabla : M \to M \otimes_R \Omega^2_R$ given by $K_\nabla = \nabla^1 \circ \nabla$, where ∇^1 is the natural extension of ∇ to $M \otimes_R \Omega_R$, and ∇ is an *integrable connection* if $K_\nabla = 0$. To avoid confusion, we shall refer to connections in the sense of Andre Weil as Ω-connections in this paper.

Lemma 9.1. *Let R be a regular local analytic $\mathbb{C}$-algebra, and let M be a finitely generated R-module. If there is an Ω-connection on M, then M is free.*

Lemma 9.2. *There is a natural functor $\Omega\mathsf{MC}(R) \to \mathsf{MC}(R)$, from the category of R-modules with Ω-connection to the category of R-modules with connection, and an induced functor $\Omega\mathsf{MIC}(R) \to \mathsf{MIC}(R)$ between categories with modules with integrable connections. If Ω_R and $\mathrm{Der}_{\mathbb{C}}(R)$ are projective R-modules of finite presentation, then these functors are equivalences of categories.*

Proof. Any Ω-connection on M induces a connection on M, and this assignment preserves integrability. Moreover, any connection ∇ on M may be considered as a $\mathbb{C}$-linear map $M \to \mathrm{Hom}_R(\mathrm{Der}_{\mathbb{C}}(R), M)$, given by $m \mapsto \{D \mapsto \nabla_D(m)\}$. It is sufficient to show that the natural map $M \otimes_R \Omega_R \to \mathrm{Hom}_R(\mathrm{Der}_{\mathbb{C}}(R), M)$, given by $m \otimes \omega \mapsto \{D \mapsto \phi_D(\omega)m\}$, is an isomorphism. But this is clearly the case when Ω_R and $\mathrm{Der}_{\mathbb{C}}(R)$ are projective R-modules of finite presentation. □

We see that if R is a regular, then there is a bijective correspondence between (integrable) connections on M and (integrable) Ω-connections on M for any R-module M. In contrast, there are many modules that admit connections but not Ω-connections when R is singular.

9.2.2 Representations of the Local Fundamental Group in Dimension Two

We may always choose a representative X of a normal surface singularity (X, x), such that $X \setminus \{x\}$ is connected and $(X, x) \subset (\mathbb{C}^n, 0)$. If $\varepsilon > 0$ is small and B_ε is a ball in $\mathbb{C}^n$ of radius ε, then $L := X \cap \partial B_\varepsilon$ is a smooth, compact, connected and oriented real manifold, called the link of (X, x), see Mumford [23]. The isomorphism class of L is independent of (small) ε. We define the local fundamental group $\pi_1^{\mathrm{loc}}(X, x) := \pi_1(L)$, and we will always assume that the representative X is such that $\pi_1^{\mathrm{loc}}(X, x) = \pi_1(X \setminus \{x\})$.

Assume that M is a maximal Cohen–Macaulay R-module of rank r, represented by a sheaf $\mathscr{M}$ on X. Then $\mathscr{M}$ is locally free on $U = X \setminus \{x\}$. Assume that M admits an integrable (flat) connection ∇. Then it follows by localization that $\mathscr{M}_{|U}$ admits an integrable connection $\nabla_U : \mathscr{M}_{|U} \to \mathscr{M}_{|U} \otimes \Omega^1_U$. The kernel $\operatorname{Ker} \nabla_U$ is a local system on U and corresponds to a representation

$$\rho_{(M,\nabla)} : \pi_1^{\mathrm{loc}}(X,x) \to \mathrm{Gl}(\mathbb{C},r)$$

called the monodromy representation of (M,∇).

9.3 Obstruction Theory

Let R be a $\mathbb{C}$-algebra. For any R-modules M, M', we refer to Weibel [25] for the definition of the *Hochschild cohomology groups* $\mathrm{HH}^n(R, \mathrm{Hom}_{\mathbb{C}}(M,M'))$ of R with values in the bimodule $\mathrm{Hom}_{\mathbb{C}}(M,M')$, and we recall that there is a natural isomorphism of $\mathbb{C}$-linear vector spaces $\mathrm{Ext}^n_R(M,M') \to \mathrm{HH}^n(R, \mathrm{Hom}_{\mathbb{C}}(M,M'))$ for any $n \geq 0$. Note that there is an exact sequence of $\mathbb{C}$-linear vector spaces

$$0 \to \mathrm{HH}^0(R, \mathrm{Hom}_{\mathbb{C}}(M,M')) = \mathrm{Hom}_R(M,M') \to \mathrm{Hom}_{\mathbb{C}}(M,M') \xrightarrow{\mu} \mathrm{Der}_{\mathbb{C}}(R, \mathrm{Hom}_{\mathbb{C}}(M,M')) \to \mathrm{HH}^1(R, \mathrm{Hom}_{\mathbb{C}}(M,M')) \to 0, \quad (9.2)$$

where μ is the assignment which maps $\phi \in \mathrm{Hom}_{\mathbb{C}}(M,M')$ to the trivial derivation given by $(a,m) \mapsto a\phi(m) - \phi(am)$ for all $a \in R$, $m \in M$.

Proposition 9.1. *There is a canonical map $g : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{Ext}^1_R(M,M)$, called the Kodaira–Spencer map of M, with the following properties:*

1. *The Kodaira–Spencer kernel $\mathsf{V}(M) = \mathrm{Ker}(g)$ is a Lie algebra and an R-module.*
2. *For any $D \in \mathrm{Der}_{\mathbb{C}}(R)$, there exists an operator $\nabla_D \in \mathrm{End}_{\mathbb{C}}(M)$ satisfying the Leibniz rule (9.1) if and only if $D \in \mathsf{V}(M)$.*

In particular, there is an action of $\mathrm{Der}_{\mathbb{C}}(R)$ on M if and only if g is trivial.

Proof. Let $\psi_D \in \mathrm{Der}_{\mathbb{C}}(R, \mathrm{End}_{\mathbb{C}}(M))$ be given by $\psi_D(a)(m) = D(a)m$ for any $D \in \mathrm{Der}_{\mathbb{C}}(R)$, and denote by $g(D)$ the class in $\mathrm{Ext}^1_{\mathbb{C}}(M,M)$ corresponding to the class of ψ_D in $\mathrm{HH}^1(R, \mathrm{End}_{\mathbb{C}}(M))$. This defines the Kodaira–Spencer map g of M, which is R-linear by definition. Clearly, its kernel $\mathsf{V}(M)$ is closed under the Lie product. Using the exact sequence (9.2), it easily follows that there exists an operator ∇_D satisfying the Leibniz rule (9.1) with respect to D if and only if $D \in \mathsf{V}(M)$. Hence, if g is trivial, we may choose a $\mathbb{C}$-linear operator ∇_D with this property for each derivation $D \in \mathrm{Der}_k(R)$ in a $\mathbb{C}$-linear basis for $\mathrm{Der}_{\mathbb{C}}(R)$, and we obtain an action $\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M)$ of $\mathrm{Der}_{\mathbb{C}}(R)$ on M. One the other hand, if there is an action of $\mathrm{Der}_{\mathbb{C}}(R)$ on M, we must have $\mathsf{V}(M) = \mathrm{Der}_{\mathbb{C}}(R)$. □

We remark that the Kodaira–Spencer map g is the contraction against the first Atiyah class of the R-module M. See Källström [19], Sect. 2.2 for another proof of Proposition 9.1. Some useful properties of the Kodaira–Spencer kernel may be found in Lemma 5 in Eriksen and Gustavsen [10]. See also Buchweitz and Liu, [4], Lemma 3.4.

Proposition 9.2. *Assume that the Kodaira–Spencer map of M is trivial. Then there is a canonical class* $\mathsf{lc}(M) \in \mathrm{Ext}^1_R(\mathrm{Der}_{\mathbb{C}}(R), \mathrm{End}_R(M))$ *such that* $\mathsf{lc}(M) = 0$ *if and only if there exists a connection on M. Moreover, if* $\mathsf{lc}(M) = 0$*, then there is a transitive and effective action of* $\mathrm{Hom}_R(\mathrm{Der}_{\mathbb{C}}(R), \mathrm{End}_R(M))$ *on the set of connections on M.*

Proof. Choose a $\mathbb{C}$-linear connection $\nabla : \mathrm{Der}_{\mathbb{C}}(R) \to \mathrm{End}_{\mathbb{C}}(M)$ on M, and let $\phi \in \mathrm{Der}_{\mathbb{C}}(R, \mathrm{Hom}_{\mathbb{C}}(\mathrm{Der}_{\mathbb{C}}(R), \mathrm{End}_R(M)))$ be the derivation given by $\phi(a)(D) = a\nabla_D - \nabla_{aD}$. We denote by $\mathsf{lc}(M)$ the class in $\mathrm{Ext}^1_R(\mathrm{Der}_{\mathbb{C}}(R), \mathrm{End}_R(M))$ corresponding to the class of ϕ in $\mathrm{HH}^1(R, \mathrm{Hom}_{\mathbb{C}}(\mathrm{Der}_{\mathbb{C}}(R), \mathrm{End}_R(M)))$, and it is easy to check that this class is independent of ∇. Using the exact sequence (9.2), the proposition follows easily. □

When the Kodaira–Spencer map of M is trivial, there is a natural short exact sequence $0 \to \mathrm{End}_R(M) \to c(M) \to \mathrm{Der}_{\mathbb{C}}(R) \to 0$ of left A-modules, where $c(M) = \{\phi \in \mathrm{End}_{\mathbb{C}}(M) : [\phi, a] \in R \text{ for all } a \in R\}$ is the module of first order differential operators on M with scalar symbol. We remark that $c(M)$ represents $\mathsf{lc}(M)$ as an extension of R-modules, see also Källström [19], Proposition 2.2.10.

9.3.1 Implementation in Singular

We have implemented the obstruction theory in a library `conn.lib` [9] for the computer algebra system Singular 3.0. The implementation is explained in detail in Eriksen and Gustavsen [10] and can be used to compute many interesting examples. Some examples are given in Sect. 9.4.

We have used Hochschild cohomology to define this obstruction theory. However, the description of the obstructions in terms of free resolutions is essential for the implementation; see Sect. 4 of Eriksen and Gustavsen [10].

In the case of simple hypersurface singularities (of type A_n, D_n and E_n) in dimension d, there exists a connection on any MCM module if $d \leq 2$. Using our implementation, we got interesting results in higher dimensions: For $d = 3$, we found that the only MCM modules that admit connections are the free modules if $n \leq 50$, and experimental results indicated that the same result should hold for $d = 4$. Using different techniques, we proved a more general result in [11]: An MCM module over a simple hypersurface singularity of dimension $d \geq 3$ admits a connection if and only if it is free.

9.4 Results and Examples

In this section, we explain some results on existence of connections. The main examples are the so-called CM-finite and CM-tame singularities. We say that (X,x) or $R = \mathscr{O}_{X,x}$ is CM-finite if there are only finitely many indecomposable MCM R-modules. We say that (X,x) or $R = \mathscr{O}_{X,x}$ is CM-tame if it is not CM-finite and if there are at most a finite number of 1-parameter families of indecomposable MCM R-modules, see [6].

Theorem 9.1 (Knörrer [21], Buchweitz–Greuel–Schreyer [5]). *A hypersurface* $(X,x) = (V(f),0) \subseteq (\mathbb{C}^{d+1},0)$ *is CM-finite if and only if it is simple, i.e. f is one of the following:*

$$\begin{aligned}
A_n &: f = x^2 + y^{n+1} + z_1^2 + \cdots + z_{d-1}^2 && n \geq 1\\
D_n &: f = x^2y + y^{n-1} + z_1^2 + \cdots + z_{d-1}^2 && n \geq 4\\
E_6 &: f = x^3 + y^4 + z_1^2 + \cdots + z_{d-1}^2\\
E_7 &: f = x^3 + xy^3 + z_1^2 + \cdots + z_{d-1}^2\\
E_8 &: f = x^3 + y^5 + z_1^2 + \cdots + z_{d-1}^2
\end{aligned}$$

9.4.1 Dimension One

In this section we consider curve singularities. We have the following:

Theorem 9.2 (Eriksen [8]). *If R is the local ring of a simple curve singularity and M is an MCM R-module, then there is an integrable connection on M.*

Let R be the local ring of any curve singularity. We say that a local ring S *birationally dominates* S if $R \subseteq S \subseteq R^*$, where R^* is the integral closure of R in its total quotient ring. It is known that R has finite CM representation type if and only if it birationally dominates the complete local ring of a simple curve singularity, see Greuel and Knörrer [13].

This result leads to a complete classification of curve singularities of finite CM representation type. The only Gorenstein curve singularities of finite CM type are the simple singularities, and the non-Gorenstein curve singularities of finite CM representation type are of the following form:

$$\begin{aligned}
D_n^s &: R = \mathbb{C}\{x,y,z\}/(x^2 - y^n, xz, yz) \text{ for } n \geq 2\\
E_6^s &: R = \mathbb{C}\{t^3,t^4,t^5\} \subseteq \mathbb{C}\{t\}\\
E_7^s &: R = \mathbb{C}\{x,y,z\}/(x^3 - y^4, xz - y^2, y^2z - x^2, yz^2 - xy)\\
E_8^s &: R = \mathbb{C}\{t^3,t^5,t^7\} \subseteq \mathbb{C}\{t\}
\end{aligned}$$

Using SINGULAR 3.0 [14] and our library CONN.LIB [9], we show that not all MCM R-modules admit connections in these cases. In fact, the canonical module ω_R does not admit a connection when R is the local ring of the singularities E_6^s, E_7^s, E_8^s or D_n^s for $n \leq 100$.

Theorem 9.3. *Let R be the local ring of a monomial curve singularity. Then all formally gradable MCM R-modules of rank one admits a connection if and only if R is Gorenstein.*

Proof. See Theorem 13 in Eriksen and Gustavsen [11]. □

Let us consider the non-Gorenstein monomial curve singularities E_6^s and E_8^s of finite CM representation type. By Yoshino [26], Theorem 15.14, all MCM R-modules are formally gradable in these cases. For E_6^s, we have three non-free rank one MCM R-modules M_1, M_2, M_{12}. One can show that M_2 and M_{12} admit connections, while M_1 does not, and that M_1 is the canonical module in this case. A similar consideration for E_8^s shows that M_{14}, M_2, M_4, M_{24} and M_{124} are the non-free rank one MCM A-modules, and that the canonical module M_2 is the only rank one MCM A-module that does not admit connections.

Finally, we remark that any connection on an MCM R-module is integrable when R is a monomial curve singularity.

9.4.2 Dimension Two

In dimension two, the situation is somewhat different. Notice that a Cohen–Macaulay isolated surface singularity is normal.

Theorem 9.4 (Herzog [17], Auslander [1], Esnault [12]). *Let (X,x) be a normal surface singularity. Then (X,x) is CM-finite if and only if it is a quotient singularity with local ring $R = \mathbb{C}\{u,v\}^G$ for a finite group $G = \pi_1^{\text{loc}}(X,x)$, and in this case there is a one-to-one correspondence between representations of $\pi_1^{\text{loc}}(X,x)$ and MCM R-modules.*

Proposition 9.3. *Any MCM module on a quotient surface singularity admits an integrable connection. In particular; any MCM module on a CM-finite surface singularity admits an integrable connection.*

Proof. Let $S = \mathbb{C}\{u,v\}$, and $M = (S \otimes_{\mathbb{C}} V)^G$. There is a canonical integrable connection $\nabla' : \mathrm{Der}_k(S) \to \mathrm{End}_k(S \otimes_k V)$ on the free S-module $S \otimes_k V$, given by

$$\nabla'_D(\sum s_i \otimes v_i) = \sum D(s_i) \otimes v_i$$

for any $D \in \mathrm{Der}_k(S)$, $s_i \in S$, $v_i \in V$. But the natural map $\mathrm{Der}_k(S)^G \to \mathrm{Der}_k(S^G)$ is an isomorphism, see Schlessinger [24], hence ∇' induces an integrable connection ∇ on M. □

Theorem 9.5 (Gustavsen and Ile [16]). *Let (X,x) be a normal surface singularity with $R = \mathscr{O}_{X,x}$. Then the monodromy representation gives an equivalence of categories between the category of MCM R-modules with integrable connection and horizontal (i.e. compatible) morphisms, and the category of finite-dimensional representations of the local fundamental group $\pi_1^{\text{loc}}(X,x)$.*

We now consider the tame case. Any MCM module over a simple elliptic surface singularity admits an integrable connection, see Kahn [18], and this result has been generalized to quotients of simple elliptic surface singularities in Gustavsen and Ile [15]. Kurt Behnke has pointed out that it might be true for cusp singularities as well, see Behnke [3]. More generally, it is probable that any MCM module over a log canonical surface singularity admits an integrable connection.

When R is a surface singularity, we have not found any examples of an MCM R-module that does not admit an integrable connection.

We consider the following example. Let $Y = \text{Specan}(S)$ where $S = \mathbb{C}[x,y,z]/(x^3 + y^3 + z^3)$, and let $E = \text{Projan}(S)$. Let $G = \mathbb{Z}/(3)$, and let ω be a fixed primitive third root of unity. We fix the action of G on S given by $(x,y,z) \mapsto (\omega x, \omega^2 y, \omega z)$ and we let X be the quotient of Y under this action. (Note that the action on S is compatible with the grading. In particular we have an action on E.)

The quotient $X = Y/G$ is an elliptic quotient and Y is its canonical covering. One finds that there are 27 rank one and 54 rank two MCM modules on X. There is a one parameter family of non-isomorphic rank three MCM modules on X.

Using Mumford [23], one can show that the local fundamental group of the quotient (X,x) is the group given by two generators a_1 and a_2 and the relations $a_1^3 = a_2^3 = (a_1^2 a_2^2)^3$. The representations of rank one and two are found in an appendix of Gustavsen and Ile [15]. There are 27 rank one and 54 rank two indecomposable modules. Thus in rank one and two, there is a one-to-one correspondence between indecomposable representations of the local fundamental group and indecomposable MCM modules. In particular; rank one and rank two MCM modules admit *unique integrable connections.* On (Y,y), in contrast, there is a positive dimensional family of connections on each indecomposable MCM module, see Kahn [18, Theorem 6.30].

9.4.3 Connections on MCM Modules in Dimension Three

In larger dimensions things are different:

Theorem 9.6 (Eriksen and Gustavsen [11]). *Let R be the analytic local ring of a simple singularity of dimension $d \geq 3$. Then an MCM R-module M admits a connection if and only if it is free.*

Note that if R is Gorenstein and of finite CM representation type, then R is a simple singularity, see Herzog [17]. The classification of non-Gorenstein singularities of finite CM representation type is not known in dimension $d \geq 3$. However, partial results are given in Eisenbud and Herzog [7], Auslander and Reiten [2] and Yoshino [26].

In Auslander and Reiten [2], it was shown that there is only one quotient singularity of dimension $d \geq 3$ with finite CM representation type, the cyclic threefold quotient singularity of type $\frac{1}{2}(1,1,1)$. Its local ring $R = S^G$ is the invariant ring of the action of the group $G = \mathbf{Z}_2$ on $S = \mathbb{C}\{x_1, x_2, x_3\}$ given by $\sigma x_i = -x_i$ for $i = 1,2,3$, where $\sigma \in G$ is the non-trivial element. There are exactly two non-free indecomposable MCM R-modules M_1 and M_2. The module M_1 has rank one, and is induced by the non-trivial representation of G of dimension one, see Auslander and Reiten [2]. It follows that M_1 admits an integrable connection, and one may show that M_1 is the canonical module of A.

It is also known that the threefold scroll of type $(2,1)$, with local ring

$$R = \mathbb{C}\{x,y,z,u,v\}/(xz - y^2, xv - yu, yv - zu),$$

has finite CM representation type, see Auslander and Reiten [2]. There are four non-free indecomposable MCM R-modules, and none of these admit connections. In particular, the canonical module ω_R does not admit a connection.

To the best of our knowledge, no other examples of singularities of finite CM representation type are known in dimension $d \geq 3$.

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