	Emne	Lærebok	Oppgaver
1	Mer om inverse matriser	[E] 6.7	[E] 6.7.4 - 6.7.5
2	Oppgaveregning: Eksamen 05/2023		Oppgave 1, 3

1 Inverse matriser

Defn: A^{-1} er løsningen X av $AX = I$ og $X \cdot A = I$

A $m\times n$-matrise

$m \neq n$: A ikke invertibel (A^{-1} fins ikke)

$m = n$:
$$\begin{cases} |A| \neq 0: & A^{-1} = \frac{1}{|A|}\cdot \operatorname{adj}(A) = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn}\end{pmatrix}^T \\ |A| = 0: & A \text{ ikke invertibel } (A^{-1} \text{ fins ikke}) \end{cases}$$

Eks: $A = \begin{pmatrix} 1 & 5 \\ -1 & 7 \end{pmatrix}$ $\quad |A| = 1\cdot 7 - 5(-1) = 12 \neq 0$ $\quad A$ invertibel

$$A^{-1} = \frac{1}{12}\cdot\begin{pmatrix} 7 & -5 \\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$$

Eks: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9\end{pmatrix}$

$$|A| = +1(18-12) - 1(9-4) + 1(3-2) = 6-5+1 = 2 \neq 0$$

A invertibel

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33}\end{pmatrix}^T$$

$C_{11} = +6 \quad C_{12} = -5 \quad C_{13} = +1$

$C_{21} = -6 \quad C_{22} = +8 \quad C_{23} = -2$

$\left(+\begin{vmatrix} 1 & 1 \\ 1 & 9\end{vmatrix}\right) \quad \left(-\begin{vmatrix} 1 & 1 \\ 1 & 3\end{vmatrix}\right)$

$C_{31} = +2 \quad C_{32} = -3 \quad C_{33} = +1$

$$= \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1\end{pmatrix}^T$$

$$= \frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1\end{pmatrix}$$

Eks: Vi har et 17×17 lineært system, som skrives $A\underline{x} = \underline{b}$ på matriseform. Anta at $|A| \neq 0$.

Da har vi: i) A^{-1} eksisterer (A invertibel)

ii)
$$A\underline{x} = \underline{b} \quad | A^{-1}\cdot$$
$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$
$$I\underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b}$$

én entydig løsn. Vi har formel for løsn.

Eks:
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$|A| = +2(3) - 1(1) + 1(-1) = 4 \neq 0$

A er invertibel (A^{-1} fins)

A symmetrisk $\Rightarrow$ $C = (c_{ij})$ symmetrisk og $C^T = C$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}^T$$

$c_{11} = 3$ $\quad c_{12} = -1$ $\quad c_{13} = -1$

$c_{21} = -1$ $\quad c_{22} = +3$ $\quad c_{23} = -1$

$c_{31} = +(-1)$ $\quad c_{32} = -1$ $\quad c_{33} = +3$

$$= \underline{\underline{\frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}}}$$

Husk: A symmetrisk med $|A| \neq 0$ $\Rightarrow$ A^{-1} symmetrisk

Husk: i) A invertibel $\Leftrightarrow$ A kvadratisk med $|A| \neq 0$.

ii) Regneregel: $(A \cdot B)^{-1} = \underline{B^{-1} \cdot A^{-1}}$

$(AB) \cdot (AB)^{-1} = I \leftarrow$ Krav

$AB \cdot B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = \underline{I}$

② Eksamen 05/2023

1.

$$A = \begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 11 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$

a) Løs $A\underline{x} = \underline{b}$:

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 2 & 1 & 2 & 0 \end{array}\right) \begin{array}{l} -2 \\ -4 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 6 & -11 & -14 & -44 \end{array}\right) -2$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 0 & -1 & 2 & -4 \end{array}\right) \qquad \begin{aligned} \underline{x} - y + 3z + 4w &= 11 \\ \underline{3y} - 5z - 8w &= -20 \\ -\underline{z} + 2w &= -4 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & w \end{array} \qquad w \text{ fri}$$

$-z = -4 - 2w \qquad z = \underline{4 + 2w}$

$3y = -20 + 8w + 5(4 + 2w) = 18w \qquad y = \underline{6w}$

$x = 11 - 4w - 3(4 + 2w) + 6w \qquad x = \underline{-1 - 4w}$

$(x, y, z, w) = \underline{\underline{(-1 - 4w, 6w, 4 + 2w, w)}}$ der w er fri

$$\underline{x} = \begin{pmatrix} -1 - 4w \\ 6w \\ 4 + 2w \\ w \end{pmatrix} \text{ der } w \text{ er fri}$$

$A\underline{x} = \underline{0}$:

$$\underline{x} = \begin{pmatrix} -4w \\ 6w \\ 2w \\ w \end{pmatrix} \quad w \text{ fri}$$

b) Skriv $\underline{v_3}$ som lin. komb av $\underline{v_1}, \underline{v_2}, \underline{v_4}$ der

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v_4} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

$$x\,\underline{v_1} + y\,\underline{v_2} + w \cdot \underline{v_4} = \underline{v_3}$$

Alt 1:

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & 2 & 1 \end{array}\right) \begin{matrix} \\ -2 \\ -4 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 0 & 3 & -8 & -5 \\ 0 & 6 & -14 & -11 \end{array}\right) \begin{matrix} \\ \\ -2 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & -1 & 4 & 3 \\ 0 & 3 & -8 & -5 \\ 0 & 0 & 2 & -1 \end{array}\right)$$

$$x - y + 4w = 3 \qquad x = \underline{2}$$
$$3y - 8w = -5 \qquad y = \underline{-3}$$
$$2w = -1 \qquad w = \underline{-1/2}$$

$$3y = -5 + 8(-1/2) = -9$$
$$x = 3 - 4(-1/2) + (-3) = 2$$

Konklusjon:

$$2\underline{v_1} - 3\underline{v_2} - \tfrac{1}{2}\underline{v_4} = \underline{v_3}$$

Alt 2: $x\underline{v_1} + y\underline{v_2} + w\underline{v_4} = +\underline{v_3}$

$$x\underline{v_1} + y\underline{v_2} - \underline{v_3} + w\underline{v_4} = \underline{0}$$

Fra a): $(x,y,z,w) = (-4w, 6w, \underline{2w}, w)$ med $2w = -1$, $w = -1/2$

$$= (2, -3, \underline{\underline{-1}}, -1/2)$$

$$\Rightarrow x = 2,\ y = -3;\ w = -1/2$$

3.

$$A = \begin{pmatrix} t & 2 & 4 \\ 2 & t & 4 \\ 2 & 4 & t \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) $|A| = +t(t^2-16) - 2(2t-8) + 4(8-2t)$

$= t^3 - 16t - 4t + 16 + 32 - 8t = t^3 - 28t + 48$

Alt: $= t(t+4)(t-4) - 4(t-4) - 8(t-4)$

$= (t-4)\cdot[t(t+4) - 4 - 8] = (t-4)(t^2+4t-12)$

$= (t-4)(t+6)(t-2)$

$= (t-4)(t-2)(t+6)$

b) Finn A^{-1} når $t=1$: $|A| = 1 - 28 + 48 = 21 \neq 0$ fra a)

$\Rightarrow A$ invertibel

$t=1$: $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$ $|A| = 21 \neq 0$

$$A^{-1} = \frac{1}{21}\begin{pmatrix} -15 & 6 & 6 \\ 14 & -7 & 0 \\ 4 & 4 & -3 \end{pmatrix}^T$$

$C_{11} = +(-15)$ $C_{12} = -(-6)$ $C_{13} = +6$

$C_{21} = -(-14)$ $C_{22} = +(-7)$ $C_{23} = -0$

$C_{31} = +4$ $C_{32} = -(-4)$ $C_{33} = +(-3)$

$$= \frac{1}{21}\begin{pmatrix} -15 & 14 & 4 \\ 6 & -7 & 4 \\ 6 & 0 & -3 \end{pmatrix}$$

c) Bestem t slik at $A\underline{x} = \underline{b}$ har én løsn.

Fra teori: én entydig løsning $\Leftrightarrow |A| \neq 0$

$|A| = 0$: $(t-4)(t-2)(t+6) = 0$

$t=4,\ t=2,\ t=-6$

Konkl: én løsn for alle t med $t \neq 4, 2, -6$

Alt: $|A| = 0$

$t^3 - 28t + 48 = 0 \longleftrightarrow \begin{vmatrix} t & 2 & 4 \\ 2 & t & 4 \\ 2 & 4 & t \end{vmatrix} = 0$

$\uparrow$

$t = 2$ er løsn: $R(1) = R(2)$

$t = 4$ er løsn: $R(2) = R(3)$

Prøv! $\pm 1, \pm 2, \pm 3, \pm 4, \ldots$

$t^3 - 28t + 48 =$

$\underline{(t-2)(t-4)(t+6)}$