	Emne	Lærebok	Oppgaver
1	Repetisjon og oppgavegjennomgang		[Ark] 6, 7b, 10
2	Lineære systemer og Kramers regel	[E] 6.4	[E] 6.4.6 - 6.4.7

1 Repetisjon

m×n lineært system:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\}$$

Matriseform: $A \cdot \underline{x} = \underline{b}$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

koeff.matrise

$m \neq n$: Gauss-eliminasjon

$m = n$:

i) Gauss-eliminasjon

ii) $\det(A)$: $\det(A) \neq 0 \iff$ én entydig løsning

$\det(A) = 0 \iff$ ingen løsn. eller uendelig mange løsn.

Oppgave ark 7b

$$\begin{array}{l} ax + y = 1 \\ -x + ay = 2 \end{array}$$

$$\left(\begin{array}{cc|c} a & 1 & 1 \\ -1 & a & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & a & 2 \\ a & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & a & 2 \\ 0 & 1+a^2 & 1+2a \end{array}\right)$$

én entydig løsning

$-x + ay = 2$

$(1+a^2)y = 1+2a$

$x = -2 + a \cdot \frac{1+2a}{1+a^2} \quad y = \frac{1+2a}{1+a^2}$ //

$$x = \frac{-2(1+a^2) + a(1+2a)}{1+a^2} = \frac{a-2}{1+a^2}$$ //

For alle a er det én entydig løsning

$$x = \frac{a-2}{1+a^2} \qquad y = \frac{1+2a}{1+a^2}$$

Alt: $|A| = \begin{vmatrix} a & 1 \\ -1 & a \end{vmatrix} = a^2+1$

$|A|=0$: $a^2+1=0$ $\Rightarrow$ $|A| \neq 0$ for alle a

$a^2=-1$ én entydig løsning for alle a

umulig

Eks:
$$\begin{aligned} x+2y-az &= a-1 \\ ax+2y-z &= 3 \\ x+(a+1)y-z &= 3 \end{aligned}$$

3x3 lineært system med parameter a

Alt I:
$$\left(\begin{array}{ccc|c} 1 & 2 & -a & a-1 \\ a & 2 & -1 & 3 \\ 1 & a+1 & -1 & 3 \end{array}\right) \begin{matrix} -a \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -a & a-1 \\ 0 & 2-2a & -1+a^2 & 3-a(a-1) \\ 0 & a-1 & a-1 & 3-(a-1) \end{array}\right)$$

$a=1$:

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right) \, -1 \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

ingen løsn.

$a \neq 1$:

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -a & a-1 \\ 0 & 2-2a & a^2-1 & -a^2+a+3 \\ 0 & a-1 & a-1 & -a+4 \end{array}\right) \, \tfrac{1}{2} \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -a & a-1 \\ 0 & 2-2a & a^2-1 & -a^2+a+3 \\ 0 & 0 & * & -a+4 \end{array}\right)$$

$$* = a-1+\tfrac{1}{2}(a^2-1) = (a-1)\left(1+\tfrac{1}{2}(a+1)\right)$$

$a=-3$ $a \neq -3$

Alt 2: Determinant

$$|A| = \begin{vmatrix} 1 & 2 & -a \\ a & 2 & -1 \\ 1 & a+1 & -1 \end{vmatrix} = 1\left(-2+a+1\right) - 2\left(-a+1\right) - a\left(a(a+1)-2\right)$$

$$= a-1 + \underbrace{2a-2}_{2(a-1)} - a\underbrace{(a^2+a-2)}_{a(a-1)(a+2)} = (a-1)\cdot\left[1+2 - a(a+2)\right]$$

$$= (a-1)(-a^2-2a+3) = -(a-1)(a^2+2a-3)$$
$$= -(a-1)(a+3)(a-1)$$

$|A|=0$ for $a=1$, $a=-3$

$a=1$:
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \end{array}\right) \begin{array}{l} \\ -1 \\ -1 \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$
ingen løsn.

$a=-3$:
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \end{array}\right) \begin{array}{l} \\ 3 \\ -1 \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & 8 & 8 & -9 \\ 0 & -4 & -4 & 7 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{2} \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 0 & 8 & 8 & -9 \\ 0 & 0 & 0 & \underbrace{7-9/2}_{5/2} \end{array}\right)$$
ingen løsn

$|A| \neq 0$ for $a \neq 1, -3$: én entydig løsning

$$x = \frac{|A_x(\underline{b})|}{|A|} \qquad y = \frac{|A_y(\underline{b})|}{|A|} \qquad z = \frac{|A_z(\underline{b})|}{|A|} \quad \leftarrow \text{Kramers regel}$$

$|A_x(\underline{b})|$ = determinanten til matrisen vi får når vi bytter ut x-kolonnen i A med $\underline{b}$

$$A = \begin{pmatrix} 1 & 2 & -a \\ a & 2 & -1 \\ 1 & a+1 & -1 \end{pmatrix} \qquad |A| = \underline{-(a-1)^2(a+3)}$$

$$\underline{b} = \begin{pmatrix} a-1 \\ 3 \\ 3 \end{pmatrix}$$

$$|A_x(\underline{b})| = \begin{vmatrix} a-1 & 2 & -a \\ 3 & 2 & -1 \\ 3 & a+1 & -1 \end{vmatrix} = (a-1)(-2+a+1) - \cancel{2(-3+3)} - a(3(a+1)-6)$$

$$= (a-1)\underline{(a-1)} - a(3a-3)$$

$$= (a-1)\cdot[a-1-3a]$$

$$= (a-1)(-2a-1)$$

(2) Kramers regel:

$$x = \frac{\cancel{(a-1)}(-2a-1)}{-(a-1)^{\cancel{2}}(a+3)} = \underline{\underline{\frac{2a+1}{(a-1)(a+3)}}}$$

$$|A_y(\underline{b})| = \begin{vmatrix} 1 & a-1 & -a \\ a & 3 & -1 \\ 1 & 3 & -1 \end{vmatrix} = 1\cdot(0) - (a-1)(-a+1) - a(3a-3)$$

$$= (a-1)^2 - 3a(a-1)$$

$$= (a-1)(a-1-3a) = \underline{(a-1)(-2a-1)}$$

$$y = \frac{-(a-1)(2a+1)}{-(a-1)^2(a+3)} = \underline{\underline{\frac{2a+1}{(a-1)(a+3)}}}$$

$$|A_z(\underline{b})| = \begin{vmatrix} 1 & 2 & a-1 \\ a & 2 & 3 \\ 1 & a+1 & 3 \end{vmatrix} = 1\cdot(6-3(a+1)) - 2(3a-3) + (a-1)\cdot(a(a+1)-2)$$

$$= \underbrace{3-3a}_{-3(a-1)} + \underbrace{6-6a}_{-6(a-1)} + \underline{(a-1)}(a^2+a-2)$$

$$= (a-1)(-9+a^2+a-2)$$

$$= (a-1)(a^2+a-11)$$

$$z = \frac{\cancel{(a-1)}(a^2+a-11)}{-(a-1)^{\cancel{2}}(a+3)}$$

$$= \underline{\underline{-\frac{a^2+a-11}{(a-1)(a+3)}}}$$

Eks.

$$\begin{aligned} ax + y &= 1 \\ -x + ay &= 2 \end{aligned} \qquad |A| = \begin{vmatrix} a & 1 \\ -1 & a \end{vmatrix} = a^2+1 \neq 0$$

$$\left.\begin{aligned} |A_x(\underline{b})| &= \begin{vmatrix} 1 & 1 \\ 2 & a \end{vmatrix} = a-2 \\ |A_y(\underline{b})| &= \begin{vmatrix} a & 1 \\ -1 & 2 \end{vmatrix} = 2a+1 \end{aligned}\right\} \underset{\text{Kramers regel}}{\Longrightarrow} \boxed{\begin{aligned} x &= \frac{a-2}{a^2+1} \\ y &= \frac{2a+1}{a^2+1} \end{aligned}}$$

<u>Resultat</u>: Kramers regel

Et lineært system $A\underline{x} = \underline{b}$ slik at $m = n$ og $|A| \neq 0$ har én entydig løsning

$$x_1 = \frac{|A_1(\underline{b})|}{|A|} \quad x_2 = \frac{|A_2(\underline{b})|}{|A|} \quad \dots \quad x_n = \frac{|A_n(\underline{b})|}{|A|}$$

der $A_i(\underline{b})$ er matrisen vi får ved å bytte ut i'te kolonne i A med $\underline{b}$.

Oppgaveark, 6:

$A = \begin{pmatrix} 3 & 7 \\ 4 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$

i) $|X| = \left|\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right| = \begin{vmatrix} 3 & 7 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 3 & 1 \end{vmatrix}$

$= +3 \cdot \begin{vmatrix} 6 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 3 & 1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 7 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 3 & 1 \end{vmatrix}$

$= 3 \cdot 6 \cdot \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} - 4 \cdot 7 \cdot \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = \underbrace{(3 \cdot 6 - 4 \cdot 7)}_{|A|} \cdot \underbrace{\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}}_{|B|}$

$= -10 \cdot (-1) = \underline{\underline{10}}$

Mer generelt: $\left|\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right| = |A| \cdot |D|$ hvis $C = 0$ eller $B = 0$

(kofaktor-utvikling langs første rad)

ii) $\left|\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right| = |A| \cdot |C| = -10 \cdot 4 = \underline{\underline{-40}}$

fordi