Plan:

1. Delvis integrasjon
2. Delbrøksoppspaltning
3. Integrasjon av rasjonale uttrykk

Repetisjon: Ubestemte integral

$$\int f(x)\,dx = F(x) + C \quad \text{hvis } F'(x) = f(x)$$

Integrasjonsregler:

$$\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

Integrasjonsteknikker: Substitusjon $\begin{cases} u = \dots & \text{(uttrykk i } x\text{, "kjernen")} \\ du = u'\cdot dx \end{cases}$

Opgs. 9 (Oppgaveark 17):

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}\,\frac{du}{\left(-\frac{1}{2\cancel{\sqrt{x}}}\right)}\frac{(-2)}{(-2)}$$

$$\boxed{\begin{aligned} u &= 1-\sqrt{x} \\ du &= -\frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$= \int \frac{-2e^u}{1}\,du = \int -2e^u\,du = -2e^u + C = \underline{\underline{-2e^{1-\sqrt{x}} + C}}$$

① Delvis integrasjon: teknikk for å integrere produkt

Eks: $\int x \cdot e^x \, dx \neq \frac{1}{2}x^2 \cdot e^x + C$

siden

$(\frac{1}{2}x^2 \cdot e^x)' = (u \cdot v)' = u'v + uv'$

$= x \cdot e^x + \frac{1}{2}x^2 \cdot e^x$

Delvis integrasjon: $\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx$

Eks: $\int \underset{u'}{x} \cdot \underset{v}{e^x} \, dx = \frac{1}{2}x^2 \cdot e^x - \int \frac{1}{2}x^2 \cdot e^x \, dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{array}}$$

$\int \underset{v}{x} \cdot \underset{u'}{e^x} \, dx = x e^x - \int e^x \cdot 1 \, dx$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$= xe^x - \int e^x dx = \underline{\underline{xe^x - e^x + C}}$

Bevis: $$(u\cdot v)' = u'\cdot v + u\cdot v'$$

integrerer på begge sider

$$u\cdot v = \int u'v\,dx + \int uv'\,dx$$

$$u\cdot v - \int uv'\,dx = \int u'v\,dx$$

$$\boxed{\int u'v\,dx = u\cdot v - \int uv'\,dx}$$

(Alternativ: $\int uv'\,dx = u\cdot v - \int u'v\,dx$)

Eks: $$\int \underbrace{x}_{u'}\cdot \underbrace{\ln x}_{v}\,dx = \frac{1}{2}x^2\cdot \ln x - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx$$

$$\boxed{\begin{array}{ll} u=\frac{1}{2}x^2 & v=\ln x \\ u'=x & v'=1/x \end{array}}$$

$$= \frac{1}{2}x^2\cdot\ln x - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2\cdot\ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

Eks: $\int \ln x \, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v} \, dx$

$$\int u'v\,dx = uv - \int uv'\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x \cdot \ln x - \int 1\,dx = x\ln x - x + C$$

$$x \cdot \ln x - (x + c) = x\ln x - x - c$$

$$\boxed{\int \ln x \, dx = x\ln x - x + C}$$

Formel for integralet av $\ln x$.

Eks: $\int \frac{\ln x}{x}\,dx = \int \underbrace{\frac{1}{x}}_{u'} \cdot \underbrace{\ln x}_{v} \, dx$

$$\boxed{\begin{array}{ll} u = \ln x & v = \ln x \\ u' = 1/x & v' = 1/x \end{array}}$$

$$= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x}\,dx$$

$$\int \frac{\ln x}{x}\,dx = (\ln x)^2 - \int \frac{\ln x}{x}\,dx$$

$$2\int \frac{\ln x}{x}\,dx = (\ln x)^2 + C$$

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$$

$$\int \frac{\ln x}{x}\,dx = \int \frac{u}{x}\cdot\frac{du}{1/x} = \int u\,du$$

$$\boxed{\begin{array}{c} u=\ln x \\ du=\frac{1}{x}\cdot dx \end{array}}$$

$$= \frac{1}{2}u^2 + C = \frac{1}{2}\cdot(\ln x)^2 + C$$

Ikke alltid lett å vite om vi skal bruke delvis integrasjon eller substitusjon.

② **Delbrøksoppspaltning** Integrasjon av rasjonale funksjoner

Eks: $\int \frac{2x-1}{x^2-4}\,dx$ ← delbrøksoppspaltning

Eks: $\int \frac{2}{1-2x}\,dx = \int \frac{2}{u}\cdot\frac{du}{-2} = \int -\frac{1}{u}\,du$

$$\boxed{\begin{array}{c} u=1-2x \\ du=(-2)\,dx \end{array}}$$

$$= -\ln|u| + C$$

$$= -\ln|1-2x| + C$$

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}} = \int \frac{1}{u}\,du$$

$$\boxed{\begin{array}{c} u=1+x^2 \\ du=2x\,dx \end{array}}$$

$$= \ln|u| + C$$

$$= \ln(1+x^2) + C$$

Delbrøksoppspalting:

$$\frac{2}{1-x} + \frac{3}{1+x} = \frac{2}{(1-x)(1+x)} \cdot (1+x) + \frac{3}{(1+x)(1-x)} \cdot (1-x)$$

$$= \frac{2(1+x)+3(1-x)}{(1-x)(1+x)}$$

$$= \frac{5-x}{1-x^2}$$

$$\int \frac{5-x}{1-x^2}\,dx = \int \frac{2}{1-x} + \frac{3}{1+x}\,dx$$

$$= -2 \cdot \ln|1-x| + 3 \cdot \ln|1+x| + C$$

$$\int \frac{2}{1-x}\,dx = \int \frac{2}{u} \cdot \frac{du}{-1} = -2\ln|u| + C$$

$$\boxed{\begin{array}{l} u = 1-x \\ du = (-1)\,dx \end{array}}$$

$$= -2\ln|1-x| + C$$

Eks: $\dfrac{2x-1}{x^2-4}$

$\int \dfrac{2x-1}{x^2-4}\,dx$

① Faktorisere x^2-4:

$x^2-4 = (x+2)(x-2)$

eller

$x^2-4=0$

$x^2=4$

$x=\pm 2$ $\Rightarrow$ $x^2-4 = (x-2)(x+2)$

② Finne A og B (konst.) slik at:

$$\frac{2x-1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$$

$\Updownarrow$

$$\frac{2x-1}{x^2-4} = \frac{A\cdot(x-2)}{(x+2)(x-2)} + \frac{B\cdot(x+2)}{(x-2)(x+2)}$$

$$\frac{2x-1}{x^2-4} = \frac{Ax-2A+Bx+2B}{(x+2)(x-2)}$$

$$2x-1 = (A+B)x + (-2A+2B)$$

$A+B=2$ $\quad$ $B=2-A$

$-2A+2B=-1$ $\quad \Downarrow$

$-2A+2(2-A)=-1$

$-4A+4=-1$

$-4A=-5$

$B=2-\frac{5}{4}=3/4$ $\quad$ $A=\frac{-5}{-4}=5/4$

$$\frac{2x-1}{x^2-4} = \frac{5/4}{x+2} + \frac{3/4}{x-2} \Leftarrow$$

$$\int \frac{2x-1}{x^2-4}\,dx = \int \frac{5/4}{x+2} + \frac{3/4}{x-2}\,dx$$

$$= \frac{5}{4}\int \frac{1}{x+2}\,dx + \frac{3}{4}\cdot\int \frac{1}{x-2}\,dx$$

$$= \underline{\underline{\frac{5}{4}\ln|x+2| + \frac{3}{4}\cdot\ln|x-2| + C}}$$

__Eks:__ $\int \frac{1}{x^2-3x+2}\,dx \quad \cancel{= \ln|x^2-3x+2| + C}$

$$\frac{1}{x^2-3x+2} = \frac{A^{=1}}{x-2} + \frac{B^{=-1}}{x-1} \quad \Big|\cdot(x-2)(x-1)$$

$x^2-3x+2 = \underline{(x-2)(x-1)}$

$x^2-3x+2=0$

$x=2,\ x=1$

$$1 = A(x-1) + B\cdot(x-2)$$

↓ Sammenløsning av koeff.

$$1 = (A+B)x + (-A-2B)$$

$A+B=0 \qquad B=-A$

$-A-2B=1 \qquad -A-2(-A)=1$

$\underline{A=1}$

$\underline{B=-1}$

innsetting

$x=1:\ 1 = A\cdot 0 + B\cdot(-1)$

$x=2:\ 1 = A\cdot 1 + B\cdot 0$

$\underline{B=-1} \quad \underline{A=1}$

$$\int \frac{1}{x^2-3x+2}\,dx = \int \frac{1}{x-2} + \frac{-1}{x-1}\,dx$$

$\ln a - \ln b = \ln(a/b)$

$$= \underline{\underline{\ln|x-2| - \ln|x-1| + C}} = \ln\frac{|x-2|}{|x-1|} + C$$

Når nevneren er et kvadrat:

Eks: $\int \frac{x}{x^2-2x+1}\,dx$

$x^2-2x+1 = (x-1)^2$

$$\cancel{\frac{x}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{x-1} = \frac{A+B}{x-1}}$$

$$\frac{x}{x^2-2x+1} = \frac{\overset{=1}{A}}{x-1} + \frac{\overset{=1}{B}}{(x-1)^2} \quad \Big| \cdot (x-1)^2$$

$$x = A\cdot(x-1) + B$$

$$= Ax + (-A+B)$$

$$\begin{aligned} A &= 1 \\ -A+B &= 0 \end{aligned} \quad \underline{B=1}$$

$$\int \frac{1}{(x-1)^2}\,dx = \qquad \begin{aligned} u &= x-1 \\ du &= dx \end{aligned}$$

$$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$= \frac{1}{-1}\cdot u^{-1} + C = \underline{\underline{-\frac{1}{x-1} + C}}$$

$$\int \frac{x}{x^2-2x+1}\,dx = \int \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \underline{\underline{\ln|x-1| - \frac{1}{x-1} + C}}$$

Eks:

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

den omvendte funksjonen til $\tan(x) = \sin(x)/\cos(x)$

$1+x^2 = \ldots$

$1+x^2 = 0$

$x^2 = -1$

ingen løsn.

3 Integrasjon av rasjonale funksjoner

$$\int \frac{f(x)}{g(x)}\,dx$$

$\leftarrow$ hvis graden til $f \geq$ graden til g, bruker vi polynomdivisjon

Eks:

$$\int \frac{x^2}{1-x}\,dx$$

$$= \int \underbrace{-x-1} + \frac{1}{1-x}\,dx$$

$$= -\frac{1}{2}x^2 - x + \int \frac{1}{1-x}\,dx$$

$$= -\frac{1}{2}x^2 - x + -\ln|1-x| + C$$

$x^2 : (-x+1) = -x-1$

$-(x^2-x)$

x

$-(x-1)$

1

$$\frac{x^2}{1-x} = -x-1+\frac{1}{1-x}$$

Eks:
$$\int \frac{x^2}{1-x^2}\,dx$$

$$x^2 : (-x^2+1) = \underline{-1}$$
$$\frac{-(x^2-1)}{1}$$

$$= \int \underline{-1} + \frac{1}{1-x^2}\,dx$$

$$= -x + \int \frac{1}{1-x^2}\,dx$$

$$\uparrow$$

delbrøkoppsp.
$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$