NORWEGIAN BUSINESS SCHOOL

Solutions:	**GRA 60353** **Mathematics**	
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Answer sheets:	Squares	
	Counts 80% of GRA 6035	The subquestions are weighted equally
		Responsible department: Economics

Question 1.

(a) We compute the determinant of A using cofactor expansion along the first column, and find that

$$\det(A) = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 4 \cdot 15 - 1 \cdot 3 + 1 \cdot (-3) = \mathbf{54}$$

Since $\det(A) \neq 0$, we have that $\text{rk}(A) = \mathbf{3}$.

(b) The characteristic equation of A is given by

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

We see that $\lambda = 3$ is a solution, so $\lambda = 3$ is an eigenvalue of A. To find all eigenvalues, we develope the determinant along the first column, and get

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)((4-\lambda)^2 - 1) - 1(3-\lambda) + 1(\lambda - 3)$$

Since we know that $\lambda = 3$ is an eigenvalue, we compute the first part of the expression and factorize with $(\lambda - 3)$ as a factor:

$$(4-\lambda)(\lambda^2 - 8\lambda + 15) + 2(\lambda - 3) = (4-\lambda)(\lambda-3)(\lambda-5) + 2(\lambda-3) = 0$$

This gives $\lambda = 3$ or $(4-\lambda)(\lambda-5) + 2 = -\lambda^2 + 9\lambda - 18 = 0$, so the eigenvalues of A are $\lambda_1 = \mathbf{3}, \lambda_2 = \mathbf{3}, \lambda_3 = \mathbf{6}$. Alternatively, we may use some elementary row operations to simplify this determinant:

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & \lambda-3 & 1-(4-\lambda)^2 \\ 0 & 3-\lambda & -3+\lambda \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the first column gives $1 \cdot \big((\lambda-3)^2 - (3-\lambda)(1-(4-\lambda)^2)\big) = 0$, or

$$(\lambda-3)(\lambda - 3 + 1 - 16 + 8\lambda - \lambda^2) = (\lambda-3)(-\lambda^2 + 9\lambda - 18) = 0$$

Also using this method, we find eigenvalues $\lambda = 3$ (with multiplicity two) and $\lambda = 6$. The matrix A **is diagonalizable** since it is symmetric.

Question 2.

(a) We compute the partial derivatives and the Hessian matrix of f:

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} 4hx^3 + 8x - (6+h)y \\ 4y^3 - (6+h)x + 8y \end{pmatrix}, \quad f'' = \begin{pmatrix} 12hx^2 + 8 & -(6+h) \\ -(6+h) & 12y^2 + 8 \end{pmatrix}$$

We see that the leading principal minors are given by $D_1 = 12hx^2 + 8$ and $D_2 = 144hx^2y^2 + 96hx^2 + 96y^2 + 64 - (6+h)^2$. Hence $D_1 \geq 0$ for all (x, y) if and only if $h \geq 0$. Moreover, if $h \geq 0$, then $D_2 \geq 0$ for all (x, y) if and only if $64 - (6+h)^2 \geq 0$, which means that $h \leq 2$. Since we can have $D_2 = 0$ (for $h = 2$), we also check the remaining principal minor $\Delta_1 = 12y^2 + 8 \geq 0$. We conclude that f is convex if and only if $0 \leq h \leq 2$. In particular, f is convex if $h = 0$.

(b) When $h = 0$, f is convex and therefore a point (x, y) is a global minimum if and only if it is a stationary point. We compute the stationary points, which are given by the equations

$$8x - 6y = 0, \quad 4y^3 - 6x + 8y = 0$$

The first equation gives that $x = 3y/4$, and the second equation becomes

$$4y^3 - 9y/2 + 8y = y(4y^2 + 7/2) = 0 \quad \Leftrightarrow \quad y = 0$$

since $4y^2 + 7/2 = 0$ has no solutions. The stationary points are therefore given by $(x, y) = (\mathbf{0}, \mathbf{0})$ when $h = 0$, and this is the global minimum, with minimum value $f(0, 0; 0) = 0$.

(c) Let $h = 0$. By the Envelope Theorem, we have that

$$\frac{d}{dh} f^*(h) = \frac{\partial f}{\partial h}\Big|_{(x,y)=(0,0)} = (x^4 - xy - 3)\big|_{(x,y)=(0,0)} = -3 < 0$$

Since the derivative is negative, the minimum value will **decrease** when h increases from $h = 0$ to small positive values of h.

Question 3.

(a) The homogeneous equation $y_{t+2} - 5y_{t+1} + 4y_t = 0$ has characteristic equation $r^2 - 5r + 4 = 0$, and therefore roots $r = 1, 4$. Hence the homogeneous solution is $y_h(t) = C_1 1^t + C_2 \cdot 4^t = C_1 + C_2 \cdot 4^t$. To find a particular solution of $y_{t+2} - 5y_{t+1} + 4y_t = 2^t$, we try $y_t = A \cdot 2^t$. This gives $y_{t+1} = 2A \cdot 2^t$ and $y_{t+2} = 4A \cdot 2^t$, and substitution in the equation gives $(4A - 10A + 4A)2^t = 2^t$, or $-2A = 1$. Hence $A = -1/2$ is a solution, and $y_p(t) = -\frac{1}{2} \cdot 2^t = -2^{t-1}$ is a particular solution. This gives general solution

$$y_t = \mathbf{C_1} + \mathbf{C_2} \cdot \mathbf{4^t} - \mathbf{2^{t-1}}$$

(b) The differential equation $y' = t(y-1)^2$ is separable, and can be written as

$$\frac{1}{(y-1)^2} y' = t \quad \Rightarrow \quad -\frac{1}{y-1} = \frac{t^2}{2} + C$$

The initial condition $y(0) = 3$ gives $-1/2 = C$. To write the solution in explicit form, we see that

$$\frac{1}{y-1} = -\frac{t^2}{2} + \frac{1}{2} = -\frac{1}{2}(t^2 - 1) \quad \Rightarrow \quad y - 1 = -\frac{2}{t^2 - 1} = \frac{2}{1 - t^2}$$

This gives the solution

$$y = 1 + \frac{2}{1 - t^2} = \mathbf{\frac{3 - t^2}{1 - t^2}}$$

(c) The differential equation $(2y-e^t)y' = ye^t + 2e^{2t}$ can be written as $-ye^t - 2e^{2t} + (2y-e^t)y' = 0$. We look for a function $h(t,y)$ such that the differential equation has the form

$$\partial h/\partial t + \partial h/\partial y \cdot y' = 0 \quad \Rightarrow \quad \begin{cases} \partial h/\partial t = -ye^t - 2e^{2t} \\ \partial h/\partial y = 2y - e^t \end{cases}$$

From the first equation, we see that $h = -ye^t - e^{2t} + C(y)$, and derivation with respect to y gives $-e^t + C'(y) = 2y - e^t$ by comparison with the second equation. So the equation is exact, and $C'(y) = 2y$ has solution $C(y) = y^2$, and $h(t,y) = y^2 - e^t y - e^{2t}$. The general solution of the differential equation is therefore $y^2 - e^t y - e^{2t} = C$, and the initial condition $y(0) = 2$ gives $4 - 2 - 1 = C$, or $C = 1$. Finally, the solution is

$$y^2 - e^t y - e^{2t} = 1 \quad \Rightarrow \quad y = \frac{e^t \pm \sqrt{e^2t + 4(e^{2t}+1)}}{2} = \frac{\mathbf{e^t + \sqrt{5e^{2t}+4}}}{\mathbf{2}}$$

by the abc-formula, where we choose the root satsifying $y(0) = 2$.

Question 4.

We rewrite the last three constraints as $-x, -y, -z \leq 0$, and write the Lagrangian for this problem as

$$\mathcal{L} = x + 2y + 2z - \lambda(x^2 + y^2 + z^2) + \nu_1 x + \nu_2 y + \nu_3 z$$

The Kuhn-Tucker conditions for this problem are the first order conditions

$$\begin{aligned} \mathcal{L}'_x &= 1 - \lambda \cdot 2x + \nu_1 = 0 \\ \mathcal{L}'_y &= 2 - \lambda \cdot 2y + \nu_2 = 0 \\ \mathcal{L}'_x &= 2 - \lambda \cdot 2z + \nu_3 = 0 \end{aligned}$$

the constraints $x^2 + y^2 + z^2 \leq 4$ and $x, y, z \geq 0$, and the complementary slackness conditions $\lambda \geq 0$, $\nu_1, \nu_2, \nu_3 \geq 0$ and

$$\lambda(x^2 + y^2 + z^2 - 4) = \nu_1 x = \nu_2 y = \nu_3 z = 0$$

Let us find all solutions of the Kuhn-Tucker conditions: If $\lambda = 0$, then $1 + \nu_1 = 0$ by the first FOC, and this is impossible since $\nu_1 \geq 0$. Hence $\lambda > 0$ and $x^2 + y^2 + z^2 = 4$. We solve the FOC's for x, y, z and get

$$x = \frac{1+\nu_1}{2\lambda}, \quad y = \frac{2+\nu_2}{2\lambda}, \quad z = \frac{2+\nu_3}{2\lambda}$$

In particular, $x, y, z > 0$ since $\nu_1, \nu_2, \nu_3 \geq 0$, and therefore we must have $\nu_1 = \nu_2 = \nu_3 = 0$ and $x = \frac{1}{2\lambda}$, $y = z = \frac{2}{2\lambda}$. When we substitute this in the first constraint, we get

$$\frac{1}{4\lambda^2}(1^2 + 2^2 + 2^2) = 4 \quad \Rightarrow \quad \lambda = \pm\frac{3}{4} = \frac{3}{4}$$

We conclude that there is a unique solution of the Kuhn-Tucker conditions:

$$(x, y, z; \lambda, \nu_1, \nu_2, \nu_3) = \left(\frac{2}{3}, \frac{4}{3}, \frac{4}{3}; \frac{3}{4}, 0, 0, 0\right)$$

with $f = 18/3$, and this is the candidate for maximum. We see that

$$\mathcal{L}(x, y, z; 3/4, 0, 0, 0) = x + 2y + 2z - \frac{3}{4}(x^2 + y^2 + z^2) \quad \Rightarrow \quad \mathcal{L}'' = \begin{pmatrix} -3/2 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & -3/2 \end{pmatrix}$$

Hence $\mathcal{L}$ is concave, and $(x, y, z) = (2/3, 4/3, 4/3)$ is max. Alternatively, we see that the set of admissible points is bounded, since $x^2 + y^2 + z^2 \leq 4$ gives $x, y, z \leq 2$. So there is a maximum by the Extreme Value Theorem, and the maximum must be obtained either at $(2/3, 4/3, 4/3)$ or at an admissible point where NDCQ is not satisfied. To consider the NDCQ condition, we compute the matrix of partial derivatives of the constraints, and get the matrix

$$\begin{pmatrix} 2x & 2y & 2z \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Clearly, not all four constraints can be simultaneously binding. In case one, two or three of the constraints are binding, the corresponding rows in the matrix are clearly independent (that is, the rank is equal to the number of rows), as we can see case by case. Hence the NDCQ holds for all admissible points, and the maximum is $(x, y, z) = (2/3, 4/3, 4/3)$ with $f = 18/3$.

Question 5.

When $\lambda = b - a$, the matrix $A - \lambda I$ is given by

$$A - (b-a)I = \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix}$$

It has rank one since $a \neq 0$. Therefore, the linear system $(A - (b-a)I)\mathbf{x} = \mathbf{0}$ has three degrees of freedom, and $\lambda = b - a$ is an eigenvalue of multiplicity at least three. We compute the trace of A to be $4b$. If $\lambda = b - a$ had multiplicity four, then the trace of A would be given by

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4(b - a)$$

But $4(b - a) \neq 4b$ since $a \neq 0$, so this is not possible, and it follows that $\lambda = b - a$ has multiplicity three. The fourth eigenvalue λ is given by

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 3(b - a) + \lambda = 4b \quad \Rightarrow \quad \lambda = b + 3a$$

The determianant is therefore

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = \mathbf{(b - a)^3 \cdot (b + 3a)}$$