Solutions	Midterm exam in GRA 6035 Mathematics
Date	January 07th, 2019 at 0900 - 1000

Correct answers: B-A-A-B D-A-B-D

Question 1.
Since $6 - \text{rk}(A) = 3$, we have that $\text{rk}(A) = 3$. The correct answer is alternative **B**.

Question 2.
We form the matrix A with the vectors $\mathbf{v}_1, \mathbf{v}_2$ as columns, and compute its minors:

$$A = \begin{pmatrix} 2 & 3 \\ t & 6 \\ 3 & t \end{pmatrix}$$

We have that $M_{12,12} = 12 - 3t$, hence $\text{rk}(A) = 2$ if $t \neq 4$. If $t = 4$, then $M_{13,12} = 2t - 9 = 8 - 9 \neq 0$, hence $\text{rk}(A) = 2$ also if $t = 4$. It follows that the vectors are linearly independent for all values of t. The correct answer is alternative **A**.

Question 3.
We compute the minor $M_{123,123}$ of A using cofactor expansion along the last column:

$$M_{123,123} = \begin{vmatrix} 1 & 3 & -1 \\ 1 & 1 & 1 \\ t & -1 & 5 \end{vmatrix} = 1(5+1) - 3(5-t) - 1(-1-t) = 4t - 8$$

This means that $\text{rk}(A) = 3$ for $t \neq 2$. When $t = 2$, we compute the rank by Gaussian elimination

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & 3 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & -7 & 7 & -5 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and see that $\text{rk}(A) = 3$ also when $t = 2$. Therefore, $\text{rk}(A) = 3$ for all values of t. The correct answer is alternative **A**.

Question 4.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 2 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the first row, which gives

$$(1-\lambda) \cdot \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 6\lambda + 5) = (1-\lambda)(\lambda-1)(\lambda-5) = 0$$

Therefore, $\lambda = 1$ is an eigenvalue of multiplicity two, and $\lambda = 5$ is an eigenvalue of multiplicity one. The correct answer is alternative **B**.

Question 5.
We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 1-\lambda & 0 & -s \\ 0 & 1-\lambda & 0 \\ s & 0 & 1-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the middle row, which gives

$$(1-\lambda) \cdot \begin{vmatrix} 1-\lambda & -s \\ s & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)^2 + s^2) = 0$$

If $s \neq 0$, then $(1-\lambda)^2 + s^2 > 0$, and $\lambda = 1$ is the only eigenvalue of A, of multiplicity one, and A is not diagonalizable. If $s = 0$, then A is diagonal and therefore diagonalizable. The correct answer is alternative **D**.

Question 6.
Since $\dim \text{Null}(A)$ equals the number of free variables, we have that $\dim \text{Null}(A) = 1$. The correct answer is alternative **A**.

Question 7.
The symmetric matrix of the quadratic form $f(x, y, z, w) = 5x^2 + 4xy + y^2 + 3z^2 + 2zw + w^2$ is given by

$$A = \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The leading principal minors are $D_1 = 5$, $D_2 = 5 - 4 = 1$, $D_3 = 3D_2 = 3$ and

$$D_4 = 5 \cdot 1(3-1) - 2 \cdot 2(3-1) = (5-4)(3-1) = 2$$

(we use cofactor expansion along the first row). Since $D_1, D_2 > 0$, $D_3 > 0$ and $D_4 > 0$, we have that A is positive definite. The correct answer is alternative **B**.

Question 8.
The function $f(x, y, z) = x^3 + y^3 + z^3 - 3(x + y + z)$ has first order partial derivatives and first order conditions given by

$$f'_x = 3x^2 - 3 = 0, \quad f'_y = 3y^2 - 3 = 0, \quad f'_z = 3z^2 - 3 = 0$$

This gives stationary points with $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, in total $2^3 = 8$ stationary points. The Hessian matrix of f is given by

$$H(f) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

This means that $H(f)(1, 1, 1)$ is positive definite, $H(f)(-1, -1, -1)$ is negative definite, and $H(f)(x, y, z)$ is indefinite at all other stationary points. Therefore, f has one local maximum, one local minimum and six saddle points. The correct answer is alternative **D**.