Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 12th, 2018 at 1500 - 1600

Correct answers: C-A-A-B A-D-A-D

Question 1.

Since $\text{rk}(A) = 5$, we also have $\text{rk}(A|\mathbf{b}) = 5$, and the linear system is consistent with $6 - 5 = 1$ degrees of freedom. The correct answer is alternative **C**.

Question 2.

We form the matrix A with the vectors $\mathbf{v}_1, \mathbf{v}_2$ as columns, and compute its minors:

$$A = \begin{pmatrix} t & 3 \\ 2 & 6 \\ 3 & t \\ 5 & 9+t \end{pmatrix}$$

We have that $M_{12,12} = 6t - 6$, hence $\text{rk}(A) = 2$ if $t \neq 1$. If $t = 1$, then $M_{13,12} = t^2 - 9 = -8$, hence $\text{rk}(A) = 2$ also if $t = 1$. It follows that the vectors are linearly independent for all values of t. The correct answer is alternative **A**.

Question 3.

We compute the minor $M_{123,123}$ of A using cofactor expansion along the last column:

$$M_{123,123} = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & 0 \\ t & -1 & 5 \end{vmatrix} = -1(-2-4t) + 5(4-6) = 4t - 8$$

This means that $\text{rk}(A) = 3$ for $t \neq 2$. When $t = 2$, we compute the rank by Gaussian elimination

$$A = \begin{pmatrix} 1 & 3 & -1 & 4 \\ 2 & 4 & 0 & 6 \\ 2 & -1 & 5 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & -7 & 7 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and see that $\text{rk}(A) = 3$ also when $t = 2$. Therefore, $\text{rk}(A) = 3$ for all values of t. The correct answer is alternative **A**.

Question 4.

We compute the eigenvalues of A by solving the characteristic equations $\det(A - \lambda I) = 0$, which gives

$$\begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

We compute the determinant by cofactor expansion along the middle row, which gives

$$(2-\lambda) \cdot \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 6\lambda + 8) = (2-\lambda)(\lambda-2)(\lambda-4) = 0$$

Therefore, $\lambda = 2$ is an eigenvalue of multiplicity two, and $\lambda = 4$ is an eigenvalue of multiplicity one. The correct answer is alternative **B**.

Question 5.

Since A is symmetric for all values of s, it is diagonalizable for all values of s. The correct answer is alternative **A**.

Question 6.

Eigenvectors of A for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.60 & 0.20 & 0.10 \\ 0.40 & -0.40 & 0.10 \\ 0.20 & 0.20 & -0.20 \end{pmatrix}$$

To simplify computations, we multiply the matrix by 10 and use Gaussian elimination:

$$10(A - I) = \begin{pmatrix} -6 & 2 & 1 \\ 4 & -4 & 1 \\ 2 & 2 & -2 \end{pmatrix} \quad \to \quad \begin{pmatrix} 2 & 2 & -2 \\ -6 & 2 & 1 \\ 4 & -4 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 2 & 2 & -2 \\ 0 & 8 & -5 \\ 0 & -8 & 5 \end{pmatrix} \quad \to \quad \begin{pmatrix} 2 & 2 & -2 \\ 0 & 8 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence v_3 is free, $v_2 = 5v_3/8$, and $v_1 = -5v_3/8 + v_3 = 3v_3/8$, and

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 3/8 \\ 5/8 \\ 1 \end{pmatrix}$$

The condition $v_1 + v_2 + v_3 = 1$ for a state vector gives $v_3 = 1/2$, and $v_2 = 5/16 = 0.3125$. The correct answer is alternative **D**.

Question 7.

The symmetric matrix of the quadratic form $f(x, y, z) = 3x^2 + 4xy - 4xz + 3y^2 + 4yz + 8z^2$ is given by

$$A = \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & 2 \\ -2 & 2 & 8 \end{pmatrix}$$

The leading principal minors are $D_1 = 3$, $D_2 = 9 - 4 = 5$ and $D_3 = |A| = 3(20) - 2(20) + (-2)(10) = 0$ (we use cofactor expansion along the first row). Since $D_1, D_2 > 0$ and $D_3 = 0$, we have that $\text{rk}(A) = 2$, and by the reduced rank criterion, we have that A is positive semi-definite (but not positive definite since $D_3 = 0$). The correct answer is alternative **A**.

Question 8.

The function $f(x, y, z) = 1 - (x - y + z)^4 = 1 - u^4$ with $u = x - y + z$. It has a stationary point in $(x, y, z) = (1, 1, 0)$ since $u(1, 1, 0) = 0$ and the first order partial derivatives are given by

$$f'_x = -4u^3 \cdot 1, \quad f'_y = -4u^3 \cdot (-1), \quad f'_z = -4u^3 \cdot 1$$

which are zero when $u = 0$. The Hessian matrix of f is given by

$$H(f) = -12u^3 \cdot \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

The leading principal minors are $D_1 = -12u^2 < 0$, and $D_2 = D_3 = 0$. At points where $u = 0$, we have that $H(f)$ is the zero matrix and is negative semi-definite. At all other points, $D_1 < 0$ and $D_2 = D_3 = 0$, so $H(f)$ is negative semi-definite by the reduced rank criterion. Therefore, f is concave, and $(1, 1, 0)$ is a global maximum. The correct answer is alternative **D**.