Solutions	Midterm exam in GRA 6035 Mathematics
Date	October 7th, 2016 at 1500 - 1600

Correct answers: B-D-C-A-D-D-B-D

Question 1.

Since $\operatorname{rk} A < \operatorname{rk}(A|\mathbf{b})$, the linear system is inconsistent. The correct answer is alternative **B**.

Question 2.

We form the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as columns, and compute its determinant

$$\begin{vmatrix} 4 & -4 & 2t \\ t & 3 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 2t(t-9) + 1(12+4t) = 2t^2 - 14t + 12 = 2(t^2 - 7t + 6) = 2(t-6)(t-1)$$

This shows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent when $t = 1, 6$, and linearly independent otherwise. The correct answer is alternative **D**.

Question 3.

We use elementary row operations to find an echelon form of the coefficient matrix A:

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & -1 & 1 & -3 \\ 1 & s & s+1 & 9 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -5 & -7 \\ 0 & s-2 & s-2 & 7 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -5 & -7 \\ 0 & 0 & 0 & * \end{pmatrix}$$

where $* = 7 - 7(s+2)/5 = 7(7-s)/5$. This means that there are two free variables if $s = 7$, and one free variable if $s \neq 7$. The correct answer is alternative **C**.

Question 4.

We have that $\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} A = 3$ and that $\lambda_1\lambda_2\lambda_3 = \det(A) = 3/4$. We can also compute $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = 3/2$ explicitly to see this. The correct answer is alternative **A**.

Question 5.

The eigenvalues are $\lambda = 1$ and $\lambda = s$. When $s \neq 1$, the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has one free variable while the multiplicity of $\lambda = 1$ is two. When $s = 1$, the eigenspace $(A - I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ has two free variables while the multiplicity of $\lambda = 1$ is three. In both cases, A is not diagonalizable. The correct answer is alternative **D**.

Question 6.

Eigenvectors for $\lambda = 1$ are given by the linear system $(A - I)\mathbf{x} = \mathbf{0}$, where

$$A - I = \begin{pmatrix} -0.23 & 0.19 \\ 0.23 & -0.19 \end{pmatrix}$$

Therefore, we see that $x = 19$ and $y = 23$ gives one eigenvector, and all others are multiple of this one. Multiplication by $1/42$ gives the state vector with $x = 19/42$ and $y = 23/42$. The correct answer is alternative **D**.

Question 7.

The symmetric matrix of the quadratic form $f(x_1, x_2, x_3, x_4) = 2x_1^2 + 6x_1x_4 + 4x_2^2 + 2x_2x_3 - 2x_2x_4 + 3x_3^2 + 5x_4^2$ is given by

$$A = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 4 & 1 & -1 \\ 0 & 1 & 3 & 0 \\ 3 & -1 & 0 & 5 \end{pmatrix}$$

The leading principal minors are $D_1 = 2$, $D_2 = 8$, $D_3 = 2(12-1) = 22$, $D_4 = 2(4 \cdot 15 - 1 \cdot 5 - 1 \cdot 3) - 3 \cdot 3 \cdot (12-1) = 5$. Since all leading principal minors are positive, f is positive definite. The correct answer is alternative **B**.

Question 8.

The function $f(x,y) = x \ln y - ay \ln x$ has Hessian matrix

$$H(f) = \begin{pmatrix} ay/x^2 & 1/y - a/x \\ 1/y - a/x & -x/y^2 \end{pmatrix}$$

Hence $D_1 = ay/x^2 > 0$ for $a = 1$ and $D_1 < 0$ for $a = -1$. For f to be concave, we need $D_1 \leq 0$, hence f may be concave for $a = -1$ but not for $a = 1$. For $a = -1$, we continue with

$$D_2 = \frac{xy}{x^2y^2} - \left(\frac{x+y}{xy}\right) = \frac{xy - (x+y)^2}{x^2y^2}$$

We find that $xy - (x+y)^2 = -x^2 - xy - y^2 \leq 0$ for all x, y since this is a negative definite quadratic form. Therefore, we do not have $D_2 \geq 0$ for all x, y when $a = -1$. The correct answer is alternative **D**.