Solutions	Final exam in GRA 6035 Mathematics
Date	November 24th 2025 at 0900 - 1400

Question 1.

(a) The differential equation $y'' + 4y' - 21y = 0$ is a linear second order differential equation that is homogeneous, and it has characteristic equation $r^2 + 4r - 21 = 0$. The characteristic roots are $r_1 = 3$ and $r_2 = -7$, since $3 + (-7) = -4$ and $3(-7) = -21$. Hence the general solution is $y = C_1 \cdot e^{3t} + C_2 \cdot e^{-7t}$.

(b) The Markov chain is regular since $A > 0$ is a positive matrix, and the vector $\mathbf{w} = (1, 1)$ is an eigenvector of A with eigenvalue $\lambda = 1$ since $A\mathbf{w} = \mathbf{w}$. Since $1 + 1 = 2$, the equilibrium state of the Markov chain is $\mathbf{v} = \frac{1}{2}\mathbf{w} = (1/2, 1/2)$.

(c) The quadratic form $f(x, y, z) = x^2 + 4xy + 6xz + 4y^2 + 12yz + 10z^2$ has symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \end{pmatrix}$$

The leading principal minors are $D_1 = 1$, $D_2 = 4 - 4 = 0$ and $D_3 = 0$ since the second row is two times the first row. We compute all principal minors $\Delta_1 = 1, 4, 10 \geq 0$, $\Delta_2 = 0, 4, 1 \geq 0$ and $\Delta_3 = 0$, and conclude that the quadratic form is positive semidefinite.

(d) A matrix A is orthogonal if $A^{-1} = A^T$. We know that $|A^T| = |A|$, and we express $|A^{-1}|$ in terms of $|A|$ using that

$$A \cdot A^{-1} = I \quad \Rightarrow \quad |A| \cdot |A^{-1}| = |A \cdot A^{-1}| = |I| = 1 \quad \Rightarrow \quad |A^{-1}| = 1/|A|$$

This means that $|A| = 1/|A|$, or that $|A|^2 = 1$, and it follows that $|A| = 1$ or $|A| = -1$.

Question 2.

(a) The trace of A is $\mathrm{tr}(A) = -3 + 0 + 3 = 0$, and we find the determinant of A using cofactor expansion along the first row:

$$\det(A) = \begin{vmatrix} -3 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 3 \end{vmatrix} = -3(0 + 4) - 2(-6 + 0) = -12 + 12 = 0$$

This gives $\mathrm{tr}(A) = 0$ and $\det(A) = 0$.

(b) We use Gaussian elimination to solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$:

$$A = \begin{pmatrix} -3 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 2 \\ -3 & 2 & 0 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 2 \\ 0 & 2 & -3 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that there is one free variable z, and back substitution gives $2y = 3z$, or $y = 3z/2$, and $-2x = -2z$, or $x = z$. The solutions can therefore be written as

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 3z/2 \\ z \end{pmatrix} = \frac{z}{2} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

This means that $\mathbf{v} = (2, 3, 2)$ is base of the nullspace of A.

(c) The characteristic equation of A is $-\lambda^3 + \mathrm{tr}(A)\lambda^2 - c_2\lambda + \det(A) = 0$, where $\mathrm{tr}(A) = \det(A) = 0$ from (a), and $c_2 = 4 + 4 - 9 = -1$. We obtain the equation $-\lambda^3 + \lambda = 0$. Alternatively, we could find the characteristic equation using cofactor expansion of $\det(A - \lambda I)$:

$$\begin{aligned} \begin{vmatrix} -3 - \lambda & 2 & 0 \\ -2 & -\lambda & 2 \\ 0 & -2 & 3 - \lambda \end{vmatrix} &= (-3 - \lambda)(-\lambda(3 - \lambda) + 4) - 2(-2(3 - \lambda)) \\ &= (-3 - \lambda)(\lambda^2 - 3\lambda + 4) + 4(3 - \lambda) \\ &= -3\lambda^2 + 9\lambda - 12 - \lambda^3 + 3\lambda^2 - 4\lambda + 12 - 4\lambda = -\lambda^3 + \lambda = 0 \end{aligned}$$

The characteristic equation can be written as $-\lambda(\lambda^2 - 1) = -\lambda(\lambda - 1)(\lambda + 1) = 0$, and it follows that the eigenvalues of A are $\lambda = -1, 0, 1$. Since A has three distinct eigenvalues, A is diagonalizable.

Question 3.

(a) We note that u is a quadratic function, and write $u(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C$ in matrix form with

$$A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & -8 & -16 \end{pmatrix}, \quad C = 21$$

The first order conditions for u are $2A\mathbf{x} + B^T = \mathbf{0}$, which can be written $2A\mathbf{x} = -B^T$, or $A\mathbf{x} = -B^T/2$. We solve this linear system using Gaussian elimination:

$$\left(\begin{array}{rrr|r} 3 & -1 & 2 & 3 \\ -1 & 2 & 1 & 4 \\ 2 & 1 & 4 & 8 \end{array}\right) \to \left(\begin{array}{rrr|r} -1 & 2 & 1 & 4 \\ 3 & -1 & 2 & 3 \\ 2 & 1 & 4 & 8 \end{array}\right) \to \left(\begin{array}{rrr|r} -1 & 2 & 1 & 4 \\ 0 & 5 & 5 & 15 \\ 0 & 5 & 6 & 16 \end{array}\right) \to \left(\begin{array}{rrr|r} -1 & 2 & 1 & 4 \\ 0 & 5 & 5 & 15 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Back substitution gives $z = 1$, $5y + 5 = 15$, or $y = 2$, and $-x + 4 + 1 = 4$, or $x = 1$. Hence u has a unique stationary point $(x, y, z) = (1, 2, 1)$.

(b) The leading principal minors of A are $D_1 = 3$, $D_2 = 6-1 = 5$ and $D_3 = 2(-5)-1(5)+4(5) = 5$, and it follows that A is positive definite. Since the Hessian $H(u) = 2A$ is also positive definite, u is convex, and the stationary point found in (a) is the minimum points of u. The minimum value of u is

$$u_{\min} = u(1, 2, 1) = 19 - 38 + 21 = 2$$

(c) We consider the outer function $f(u) = u/\ln(u)$ as a function of one variable with domain of definition equal to the range of u, which can be written $u \geq 2$. We compute the derivative of the outer function:

$$f'(u) = \frac{1 \cdot \ln u - u \cdot 1/u}{(\ln u)^2} = \frac{\ln u - 1}{(\ln u)^2}$$

Hence $f'(u) = 0$ when $\ln(u) - 1 = 0$, and this gives $\ln(u) = 1$, or $u = e^1 = e > 2$. We see that f is decreasing in the interval $[2, e]$ since $f'(u) < 0$, and increasing in $[e, \infty)$ since $f'(u) > 0$. This means that $u = e$ is a minimum point for f, and $f_{\min} = f(e) = e/\ln(e) = e/1 = e$. Since $u/\ln u \to \infty$ as $u \to \infty$, the range of f is $V_f = [e, \infty)$.

Question 4.

(a) We write the system of differential equations in the form $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 21 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

To solve the homogeneous equation, we find the eigenvalues and eigenvectors of A. The characteristic equation is $\lambda^2 + 4\lambda - 21 = 0$ since A has trace -4 and determinant -21. We factorize this as $(\lambda - 3)(\lambda + 7) = 0$, which gives $\lambda_1 = 3$ and $\lambda_2 = -7$; note that this is the same characteristic equation as in Question 1(a). Since each eigenvalue has multiplicity one, there is a base $\mathbf{v}_i$ for E_{λ_i} which we can find using Gaussian elimination:

$$E_3 : \begin{pmatrix} -3 & 1 \\ 21 & -7 \end{pmatrix} \to \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \qquad E_7 : \begin{pmatrix} 7 & 1 \\ 21 & 3 \end{pmatrix} \to \begin{pmatrix} 7 & 1 \\ 0 & 0 \end{pmatrix}$$

We may choose the base vectors $\mathbf{v}_1 = (1, 3)$ and $\mathbf{v}_2 = (1, -7)$. This gives

$$\mathbf{y}_h = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t} = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot e^{3t} + C_2 \begin{pmatrix} 1 \\ -7 \end{pmatrix} \cdot e^{-7t}$$

We find the equilibrium state by solving $A\mathbf{y} + \mathbf{b} = \mathbf{0}$, which is a linear system $A\mathbf{y} = -\mathbf{b}$ that we can solve using Gaussian elimination:

$$\left(\begin{array}{rr|r} 0 & 1 & -5 \\ 21 & -4 & -1 \end{array}\right) \to \left(\begin{array}{rr|r} 21 & -4 & -1 \\ 0 & 1 & -5 \end{array}\right)$$

Back substitution gives $y_2 = -5$, and $21y_1 = -1 + 4(-5) = -21$, or $y_1 = -1$. The equilibrium state is therefore $\mathbf{y}_e = (-1, -5)$, and since $\mathbf{y}_p = \mathbf{y}_e$ is a particular solution of the inhomogeneous system of differential equations $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, the general solution is given by

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot e^{3t} + C_2 \begin{pmatrix} 1 \\ -7 \end{pmatrix} \cdot e^{-7t} + \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

(b) The difference equation is linear, with general solution $y_t = y_t^h + y_t^p$, and $y_t^h = C \cdot 1^t = C$ since the characteristic equation of the homogeneous equation is $r - 1 = 0$, which gives $r = 1$. To find the particular solution, we guess that it has the form $y_t = At + B$. This gives

$$A(t+1) + B - (At + B) = 1 + 4t \quad \Rightarrow \quad A = 1 + 4t$$

This is not possible (note that $At + B$ contains a constant term, and the homogeneous solution is contant). We therefore try to multiply y_t with t, which gives $y_t = At^2 + Bt$. We substitute this into the difference equation:

$$A(t+1)^2 + B(t+1) - (At^2 + Bt) = 1 + 4t \quad \Rightarrow \quad 2At + A + B = 1 + 4t$$

By comparing coefficients, this gives $2A = 4$, or $A = 2$, and $A + B = 1$, or $B = 1 - A = -1$. It follows that the general solution of the difference equation is given by

$$y_t = y_t^h + y_t^p = C + 2t^2 - t = 2t^2 - t + C$$

(c) We try to write the differential equation $2t - 2ty + (2y - t^2)y' = 0$ in the form $h'_t + h'_y \cdot y' = 0$ for some function $h = h(t, y)$. The first equation $h'_t = 2t - 2ty$ gives $h = t^2 - t^2 y + C(y)$, and when we substitute this into the second equation $2y - t^2 = h'_y$ we get $2y - t^2 = -t^2 + C'(y)$, and clearly $C(y) = y^2$ gives a solution. It follows that the differential equation is exact, and the general solution can be written $t^2 - t^2 y + y^2 = C$. The initial condition $y(1) = 0$ gives $1 = C$, or $y^2 - t^2 y + t^2 = 1$. We write this in the form $y^2 - t^2 y + (t^2 - 1) = 0$ and find an explicit solution for y using the quadratic formula:

$$y = \frac{t^2 \pm \sqrt{t^4 - 4(t^2 - 1)}}{2} = \frac{t^2 \pm \sqrt{t^4 - 4t^2 + 4}}{2} = \frac{t^2 \pm \sqrt{(t^2 - 2)^2}}{2} = \frac{t^2 \pm (t^2 - 2)}{2}$$

Hence $y = t^2 - 1$, since the other solution $y = 1$ does not satsify $y(1) = 0$. Alternatively, it is possible to write the answer in the following form, if the expression under the square root is not simplified as $t^4 - 4t^2 + 4 = (t^2 - 2)^2$:

$$y = \frac{t^2 \pm \sqrt{t^4 - 4(t^2 - 1)}}{2} = \frac{t^2 \pm \sqrt{t^4 - 4t^2 + 4}}{2} = \frac{t^2 - \sqrt{t^4 - 4t^2 + 4}}{2}$$

One may check we need a negative sign in the front of the square root in this case, as this is the solution that satisfies the initial value $y(1) = 0$. Note that $\sqrt{t^4 - 4t^2 + 4} = |t^2 - 2| = -(t^2 - 2)$ for values of t close to $t = 1$.

Question 5.

(a) We notice that objective function is a linear form $f(\mathbf{x}) = B\mathbf{x}$ for a matrix B with one row, and that the constraint can be written $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \leq 11$, where A is the symmetric matrix of the quadratic form g. The matrices A and B are given by

$$A = \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 & 1 \end{pmatrix}$$

The Kuhn-Tucker problem is in standard form, with Lagrangian $\mathcal{L}(\mathbf{x}; \lambda) = B\mathbf{x} - \lambda(\mathbf{x}^T A\mathbf{x} - 11)$, and the first order conditions are given by $\mathcal{L}'(\mathbf{x}) = B^T - \lambda \cdot 2A\mathbf{x} = \mathbf{0}$ in matrix form. We write this as

$$2\lambda A\mathbf{x} = B^T \quad \Leftrightarrow \quad 2\lambda \cdot \begin{pmatrix} 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \cdot \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}$$

Alternatively, we could of course write $\mathcal{L}$ out without using matrix notation, and find the same linear system by partial derivations. The CSC is $\lambda \geq 0$, and $\lambda = 0$ if $g(\mathbf{x}) < 11$. If $\lambda = 0$, then the FOC is $B^T = \mathbf{0}$, which is impossible. Hence $\lambda > 0$ and $g(\mathbf{x}) = 11$ is a binding constraint. We can divide by 2λ in the FOC since $\lambda \neq 0$, and we write $t = \frac{1}{2\lambda} > 0$ to simplify notation. We solve the FOC in terms of t using Gaussian elimination:

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & t \\ 0 & 1 & 0 & 0 & 2t \\ 0 & 0 & 4 & 0 & 4t \\ -1 & 0 & 0 & 1 & t \end{array}\right) \to \left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & t \\ 0 & 1 & 0 & 0 & 2t \\ 0 & 0 & 4 & 0 & 4t \\ 0 & 0 & 0 & 2/3 & 4t/3 \end{array}\right) \to \left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & t \\ 0 & 1 & 0 & 0 & 2t \\ 0 & 0 & 4 & 0 & 4t \\ 0 & 0 & 0 & 2 & 4t \end{array}\right)$$

Back substitution gives $w = 2t$, $z = t$, $y = 2t$, and $3x = t + 2t$, or $x = t$. We put this into the binding constraint to find t:

$$3(t)^2 + (2t)^2 + 4(t)^2 + (2t)^2 - 2(t)(2t) = 11t^2 = 11$$

This gives $t^2 = 1$, and $t = 1$ (since $\lambda > 0$ gives $t > 0$), or $\lambda = 1/2$. It follows that there is one candidate point that satisfies the Kuhn-Tucker conditions: $(x, y, z, w; \lambda) = (1, 2, 1, 2; 1/2)$.

(b) We use the second order condition to show that the candidate point found in (a) is a maximum: We consider $h(\mathbf{x}) = \mathcal{L}(\mathbf{x}; 1/2)$ with Hessian

$$H(h) = -\frac{1}{2}(2A) = -A = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Its leading principal minors are $D_1 = -3$, $D_2 = 3$, $D_3 = -12$ and $D_4 = -1(1(4)) + 12 = 8$. Hence $H(h)$ is negative definite, and h is concave, and by the second order condition, the point $(x, y, z, w) = (1, 2, 1, 2)$ is a maximum point, with maximum value

$$f_{\max} = f(1, 2, 1, 2) = 1 + 2(2) + 4(1) + 2 = 11$$

and Lagrange multiplier $\lambda = 1/2$.

(c) We write the Kuhn-Tucker problem as a problem with parameter a, such that $a = 2$ is the case solved in (a) and (b), and the maximum value for $a = 1$ is the one we want to estimate:

$$\max f(x, y, z, w) = x + 2y + 4z + w \text{ when } 3x^2 + y^2 + 4z^2 + w^2 - axw \leq 11$$

It has Lagrangian $\mathcal{L}(x, y, z, w; \lambda) = x + 2y + 4z + w - \lambda(3x^2 + y^2 + 4z^2 + w^2 - axw - 11)$, and $\mathcal{L}'_a = -\lambda \cdot (-xw) = \lambda xw$. By the envelope theorem, it follows that at $a = 2$, the optimal value function $f^*(a)$ has derivative

$$\frac{df^*(a)}{da} = \mathcal{L}'_a(1, 2, 1, 2; 1/2) = \frac{1}{2} \cdot 1 \cdot 2 = 1$$

Using this, we estimate the new maximum value $f^*(1)$ to be

$$f^*(1) \approx f^*(2) + \Delta a \cdot \frac{df^*(a)}{da} = 11 + (1 - 2) \cdot 1 = 10$$