Exam	Final exam in GRA 6035 Mathematics
Date	January 7th, 2019 at 1300 - 1600

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.
We consider the matrix A given by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & -3 \end{pmatrix}$$

(a) **(6p)** Determine the definiteness of A.
(b) **(6p)** Find all eigenvectors for A with eigenvalue $\lambda = -5$, and compute $\dim E_{-5}$.
(c) **(6p)** Find the eigenvalues of A, and determine all values of r such that $\dim \text{Null}(A - rI) \geq 1$.

Question 2.

(a) **(6p)** Find the general solution of the differential equation $4y'' - 4y' - 3y = 9t$.
(b) **(6p)** Find the general solution of the differential equation $4ty' + 4y = 1$.
(c) **(6p)** Find the general solution of the following system of differential equations:

$$\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= 5y_2 \end{aligned}$$

Question 3.
We consider the function $f(x, y, z) = 3x^2 + y^2 + axy - y + 2z^4 + 8z + 12$, where a is a parameter.

(a) **(6p)** Find all stationary points of f when $a = 3$.
(b) **(6p)** Determine all values of a such that f is a convex function.
(c) **(6p)** Find $f^*(3)$, and use the envelope theorem to estimate $f^*(a)$ for values of a close to 3.

Question 4.
We consider the Lagrange problem

$$\min\ f(x, y, z, w) = 2x^2 + 2xy + 2y^2 + 3z^2 + 8zw - 3w^2 \text{ subject to } x^2 + y^2 + z^2 + w^2 = 1$$

(a) **(6p)** Write down the Lagrange conditions for this problem.
(b) **(6p)** Find all points $(x, y, z, w; \lambda)$ with $\lambda = -5$ that satisfy the Lagrange conditions.
(c) **(6p)** Solve the Lagrange problem, and find the minimum value f^*_{min}.

Question 5.
We consider the Kuhn-Tucker problem

$$\max\ f(x, y, z, w) = 2x^2 + 2xy + 2y^2 + 3z^2 + 8zw - 3w^2 \text{ subject to } x^2 + y^2 + z^2 + w^2 \leq 1$$

Extra credit (6p) Show that a point $(x, y, z, w; \lambda)$ satisfy the first order conditions of this problem if and only if $\mathbf{x} = (x, y, z, w)$ is an eigenvector of the matrix A (from Question 1) with eigenvalue λ or $\mathbf{x} = (0, 0, 0, 0)$. Use this to solve the Kuhn-Tucker problem and find its maximum value.