Solutions	Final exam in GRA 6035 Mathematics
Date	November 27th, 2018 at 0900 - 1200

Question 1.

(a) We use Gaussian elimination to find the rank of A:

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & -34 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three pivot positions, we have that $\text{rk}(A) = 3$. This means that $A \cdot \mathbf{x} = \mathbf{0}$ has $n - \text{rk}(A) = 4 - 3 = 1$ free variables. Since there is no pivot position in the third column, we may choose z as the free variable.

(b) By definition, $\text{Null}(A)$ is the set of solutions of the homogeneous linear system $A \cdot \mathbf{x} = \mathbf{0}$. By the computation in (a), an echelon form of the augmented matrix $(A|\mathbf{0})$ is given by

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 8 & 0 \\ 0 & 0 & 0 & -34 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Backwards substitution gives $-34w = 0$, or $w = 0$ from the third equation, $2y + 8w = 0$, or $y = -4w = 0$ from the second equation, and $x - z = 0$, or $x = z$ from the first equation. The solutions of $A \cdot \mathbf{x} = \mathbf{0}$ in vector form are therefore given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \\ 0 \end{pmatrix} = z \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = z \cdot \mathbf{v}, \quad \text{with } \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

This means that $\text{Null}(A) = \text{span}(\mathbf{v})$, and $\dim \text{Null}(A) = 1$ since there is one free variable z.

(c) The leading principal minors of A are $D_1 = 1$, $D_2 = 2$, and $D_3 = 2(1 - 1) = 0$ by cofactor expansion along the second row. Since $\text{rk}(A) = 3$, we have that $D_4 = 0$. We cannot use the reduced rank condition (RRC) to determine the definiteness of A since $\text{rk}(A) = 3$ and $D_3 = 0$, and we compute the principal minors of A instead. The first order principal minors of A are given by

$$\Delta_1 = 1, 2, 1, -2$$

Since there are both positive and negative principal minors of order one, it follows that A is indefinite.

Question 2.

(a) The differential equation $y'' - 12y' + 20y = 3e^{-t}$ is second order linear and we can solve it using superposition. To find the homogeneous solution y_h, we consider the homogeneous differential equation $y'' - 12y' + 20y = 0$, which has characteristic equation $r^2 - 12r + 20 = 0$, with two distinct solutions $r = 2$ and $r = 10$. Therefore, we have

$$y_h = C_1 e^{2t} + C_2 e^{10t}$$

To find the particular solution y_p, we consider the differential equation $y'' - 12y' + 20y = 3e^{-t}$ and use the method of undetermined coefficients. We start with $f(t) = 3e^{-t}$, and compute $f' = -3e^{-t}$ and $f'' = 3e^{-t}$. Based on this, we guess the solution $y = Ae^{-t}$, which gives $y' = -Ae^{-t}$ and $y'' = Ae^{-t}$. When we substitute this into the differential equation, we get

$$(Ae^{-t}) - 12(-Ae^{-t}) + 20(Ae^{-t}) = 3e^{-t}$$

or $33Ae^{-t} = 3e^{-t}$. Comparing coefficients, we get $33A = 3$, or $A = 1/11$, and

$$y_p = \frac{1}{11} e^{-t}$$

The general solution of the differential equation is therefore

$$y = y_h + y_p = C_1\, e^{2t} + C_2\, e^{10t} + \frac{1}{11}\, e^{-t}$$

(b) The system of differential equations can be written in the form $\mathbf{y}' = A \cdot \mathbf{y}$, where

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

We find the eigenvalues and eigenvectors of A: The eigenvalues are given by the characteristic equation $\det(A - \lambda I) = 0$, which can be written as $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 - 25 = 0$, and the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -5$. The eigenspaces are given by $E_5 = \mathrm{span}(\mathbf{v}_1)$ and $E_{-5} = \mathrm{span}(\mathbf{v}_2)$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

since we have that

$$A - \lambda_1 I = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$$

Since A is diagonalizable, with enough eigenvalues and eigenvectors, we have that the general solution is

$$\mathbf{y} = C_1 \mathbf{v}_1\, e^{\lambda_1 t} + C_2 \mathbf{v}_2\, e^{\lambda_2 t} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t} = \begin{pmatrix} 2C_1\, e^{5t} + C_2\, e^{-5t} \\ C_1\, e^{5t} - 2C_2\, e^{-5t} \end{pmatrix}$$

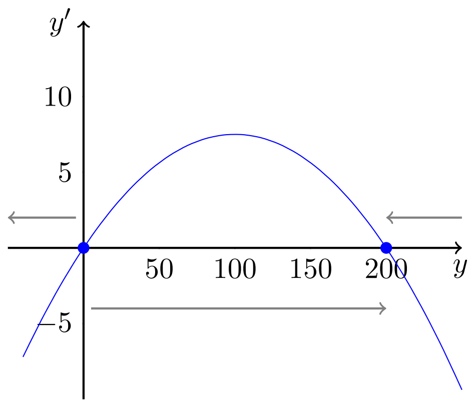
(c) The differential equation $y' = 0.15y(1 - y/200)$ is autonomous, with $F(y) = 0.15y(1 - y/200)$. The equilibrium states are therefore given by

$$F(y) = 0.15y(1 - y/200) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 200$$

Hence $y_e = 0$ and $y_e = 200$ are the equilibrium states. To determine their stability, we compute $F'(y_e)$. Since $F'(y) = (0.15y - 0.15/200y^2)' = 0.15 - 0.15y/100$, we get

$$F'(0) = 0.15 > 0, \quad F'(200) = -0.15 < 0$$

Therefore, $y_e = 0$ is unstable and $y_e = 200$ is stable by the Stability Theorem. We can also see this from the phase diagram below, where the arrows show the development of $y = y(t)$ as time passes. None of the equilibrium states are globally asymptotically stable, since an initial



value $y_0 < 0$ will give a solution curve that moves away from both equilibrium states as time passes.

Question 3.

(a) The first order partial derivatives of $f(x, y, z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 10z$ and the FOC's are given by

$$f'_x = -4x^3 - 4x + 6z = 0, \quad f'_y = -6y = 0, \quad f'_z = 6x - 12z + 10 = 0$$

The second FOC gives $y = 0$, and the third FOC gives $z = (6x + 10)/12 = (3x + 5)/6$. Substituting this in the first FOC, we get

$$-4x^3 - 4x + (3x + 5) = -4x^3 - x + 5 = 0$$

We see that $x = 1$ is a solution of this equation, and $x = 1$ gives $z = 8/6 = 4/3$. Hence there is a unique stationary point with $x = 1$, and this stationary point is $(x, y, z) = (1, 0, 4/3)$.

(b) The Hessian matrix is given by

$$H(f) = \begin{pmatrix} -12x^2 - 4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

For all x, y, z, we have that $D_1 = -12x^2 - 4 < 0$, that $D_2 = -6D_1 > 0$ and that

$$D_3 = -6(-12(-12x^2 - 4) - 36) = -6(144x^2 + 12) < 0$$

Hence $H(f)(x, y, z)$ is negative definite for all x, y, z, and f is concave. This implies that the stationary point $(x, y, z) = (1, 0, 4/3)$ is a global maximum point for f, and its global maximum value is $f_{\max} = f(1, 0, 4/3) = 71/3$.

(c) Let $f(x, y, z; a) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + az$ be a function with parameter a such that $f(x, y, z; 10) = f(x, y, z)$. From (a) and (b), we know that $f^*(10) = 71/3$ with maximum point $(x^*(10), y^*(10), z^*(10)) = (1, 0, 4/3)$. The envelope theorem states that

$$\frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \frac{\partial f(x, y, z; a)}{\partial a}\,(x^*(a), y^*(a), z^*(a))$$

when the maximum value of $f(x, y, z; a)$ exists.The right hand side is given by $z^*(a)$ since $\partial f(x, y, z; a)/\partial a = z$. For values of a close to $a = 10$, we can therefore estimate $f^*(a)$ by

$$f^*(a) \cong f^*(10) + \Delta a \cdot \frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \frac{71}{3} + (a - 10) \cdot \frac{4}{3}$$

Since $f^*(11) = \max\,(16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 11z)$, we have the following estimate of the maximum value:

$$f^*(11) \cong \frac{71}{3} + 1 \cdot \frac{4}{3} = \frac{75}{3} = 25$$

Question 4.

(a) The Lagrangian of the Kuhn-Tucker problem is $\mathcal{L} = 3x^2 - y^2 - 2z^2 - \lambda(2x^4 + 2y^4 + z^4)$ since the problem is in standard form. The first order conditions (FOC) are

$$\mathcal{L}'_x = 6x - \lambda(8x^3) = 0$$
$$\mathcal{L}'_y = -2y - \lambda(8y^3) = 0$$
$$\mathcal{L}'_z = -4z - \lambda(4z^3) = 0$$

the constraint (C) is given by $2x^4 + 2y^4 + z^4 \le 18$, and the complementary slackness conditions (CSC) are given by

$$\lambda \ge 0 \quad \text{and} \quad \lambda(2x^4 + 2y^4 + z^4 - 18) = 0$$

The Kuhn-Tucker conditions are FOC+C+CSC.

(b) The FOC's give $2x(3 - 4\lambda x^2) = 0$, $-2y(1 + 4\lambda y^2) = 0$ and $-2z(2 + 2\lambda z^2) = 0$. Since $\lambda \ge 0$, we have that

$$1 + 4\lambda y^2 > 0 \text{ and } 2 + 2\lambda z^2 > 0$$

and this implies that $y = z = 0$ from the two last FOC's. The first FOC gives $x = 0$ or $\lambda x^2 = 3/4$. If $x = 0$, then the constraint holds and is non-binding, and this implies that

$$(x, y, z; \lambda) = (0, 0, 0; 0)$$

is one candidate point with $f = 0$. Otherwise, $\lambda x^2 = 3/4$, and this implies that $\lambda > 0$, and the constraint must be binding. This gives

$$2x^4 + 0 + 0 = 18 \quad \Rightarrow \quad x^4 = 9$$

Hence $x^2 = 3$, and $x = \pm\sqrt{3}$. Since $\lambda x^2 = 3/4$, this means that $\lambda = 1/4$, and we obtain two more candidate points

$$(x, y, z; \lambda) = (\pm\sqrt{3}, 0, 0; 1/4)$$

with $f = 9$.

(c) The set of points (x, y) such that $2x^4 + 2y^4 + z^4 \leq 18$ is bounded since $x^4, y^4 \leq 9$ and $z^4 \leq 18$, which means that

$$-\sqrt{3} \leq x, y \leq \sqrt{3} \quad \text{and} \quad -\sqrt[4]{18} \leq z \leq \sqrt[4]{18}$$

It then follows from the EVT (extreme value theorem) that the Kuhn-Tucker problem has a maximum. The maximum point must be a candidate point found in (b), or an admissible point where NDCQ fails. We check the NDCQ condition: If the constraint is binding, then $2x^4 + 2y^4 + z^4 = 18$, and the NDCQ condition is

$$\operatorname{rk} J = \operatorname{rk} \begin{pmatrix} 8x^3 & 8y^3 & 4z^3 \end{pmatrix} = 1$$

The only point where this fails is $(x, y, z) = (0, 0, 0)$, and the constraint is not binding at this point. Hence NDCQ holds in the binding case. In the non-binding case, there is no condition to check. It follows that the only candidate points for maximum are the points from (b) satisfying FOC+C+CSC, and therefore the best of these candidate points are maximum points. The maximum value of f is $f_{\max} = 9$ at $(\pm\sqrt{3}, 0, 0)$.

Question 5.

The logistic differential equation $y' = 0.15y(1 - y/200)$ is separable, and can be written as

$$\frac{1}{y(1 - y/200)} \cdot y' = 0.15 \quad \Rightarrow \quad \frac{200}{y(200 - y)} \cdot y' = 0.15$$

Since the first factor can be written as $1/y + 1/(200 - y)$ using partial fractions, it follows that

$$\int \frac{1}{y} + \frac{1}{200 - y} \, \mathrm{d}y = \int 0.15 \, \mathrm{d}t \quad \Rightarrow \quad \ln|y| - \ln|200 - y| = 0.15t + C$$

Using the exponential function on both sides, we obtain

$$\frac{|y|}{|200 - y|} = e^{0.15t + C} \quad \Rightarrow \quad \frac{y}{200 - y} = \pm e^C e^{0.15t} = Ae^{0.15t}$$

This gives $y = Ae^{0.15t}(200 - y)$, or $y(1 + Ae^{0.15t}) = 200Ae^{0.15t}$, and therefore the general solution

$$y = 200 \cdot \frac{Ae^{0.15t}}{1 + Ae^{0.15t}}$$

If $y_0 = 50$, we get $50 = 200A/(1 + A)$. This means that $A/(1 + A) = 1/4$, or $4A = 1 + A$. This implies that $3A = 1$, or $A = 1/3$. The time T it takes to reach 90% of the carrying capacity $K = 200$, is given by the equation

$$0.90 \cdot 200 = 200 \cdot \frac{Ae^{0.15T}}{1 + Ae^{0.15T}} \quad \Rightarrow \quad 0.9 = \frac{Ae^{0.15T}}{1 + Ae^{0.15T}}$$

with $A = 1/3$. This gives

$$0.9(1 + Ae^{0.15T}) = Ae^{0.15T} \quad \Rightarrow \quad e^{0.15T}(A - 0.9A) = 0.9 \quad \Rightarrow \quad e^{0.15T} = \frac{0.9}{0.1A}$$

With $A = 1/3$, this gives $e^{0.15T} = 9/A = 27$, and therefore

$$0.15T = \ln(27) = \ln(3^3) = 3\ln 3 \quad \Rightarrow \quad T = \frac{3\ln 3}{0.15} = 20\ln 3$$