Solutions	Final exam in GRA 6035 Mathematics
Date	November 29th, 2017 at 0900 - 1200

Question 1.

(a) The matrix A, and an echelon form of A, is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have that $\text{rk}(A) = 2$ since the echelon form has two pivots.

(b) The leading principal minors are $D_1 = 1$, $D_2 = 1$ (by computation), and $D_3 = 0$ and $D_4 = 0$ (since A has rank two, all minors of order three and four are zero). By the RRC (reduced rank criterion), it follows from the facts that $D_1, D_2 > 0$ and that $\text{rk}(A) = 2$ that A is positive semidefinite. Therefore, f is positive semidefinite. Alternatively, we could have used the signs of all principal minors

$$\Delta_1 = 1, 1, 9, 4$$
$$\Delta_2 = 1, 9, 4, 0, 0, 0$$
$$\Delta_3 = 0, 0, 0, 0$$
$$\Delta_4 = 0$$

to come to the same conclusion.

(c) We solve $A\mathbf{x} = \mathbf{0}$ using Gaussian elimination, which gives an echelon form of the augmented matrix of the form

$$(A|\mathbf{0}) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 3 & 9 & -6 & 0 \\ 0 & -2 & -6 & 4 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Hence z, w are free variables, $x = 0$ from the first equation, and $y = -3z + 2w$ from the second equation. This gives solutions of the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -3z + 2w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = z \cdot \mathbf{v}_1 + w \cdot \mathbf{v}_2$$

Hence, the solutions of the linear system is $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$ when we put

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Note that these vectors are not unique; any choice of two linearly independent vectors that are solutions can be used instead.

Question 2.

(a) The differential equation $y' - 2y = e^t$ is linear, and we can solve it using the method of integrating factor. Since $a(t) = -2$ and

$$\int -2 \, \mathrm{d}t = -2t + C$$

it follows that $u = e^{-2t}$ is an integrating factor. We multiply with u in the differential equation, and get

$$e^{-2t}(y' - 2y) = e^{-2t} \cdot e^t \quad \Rightarrow \quad (e^{-2t} y)' = e^{-t}$$

Therefore, integration on both sides gives

$$e^{-2t}y = \int e^{-t}\,\mathrm{d}t = -e^{-t} + C \quad\Rightarrow\quad y = e^{2t}(-e^{-t} + C) = -e^t + Ce^{2t}$$

It is also possible to solve the differential equation using the superposition principle. Then $y = y_h + y_p$, and y_h is the general solution of $y' - 2y = 0$, which gives characteristic equation $r - 2 = 0$, or $r = 2$, and $y_h = Ce^{2t}$. The method of undetermined coefficients with $y = Ae^t$ gives

$$Ae^t - 2(Ae^t) = e^t \quad\Rightarrow\quad -Ae^t = e^t$$

This gives $A = -1$ and $y_p = -e^t$, and therefore $y = y_h + y_p = Ce^{2t} - e^t$.

(b) We try to solve $3t^2 - y - ty' = 0$ as an exact differential equation, and we therefore look for a function $h(t, y)$ such that

$$h'_t = 3t^2 - y, \quad h'_y = -t$$

The first condition gives $h = t^3 - ty + C(y)$ for a function $C(y)$ that is constant in t. Inserting this in the second condition, we get $-t + C'(y) = -t$. We see that we get a solution if we choose $C(y) = 0$. Therefore, the differential equation is exact and has solution

$$h = t^3 - ty = C \quad\Rightarrow\quad ty = t^3 - C \quad\Rightarrow\quad y = \frac{t^3 - C}{t} = t^2 - \frac{C}{t}$$

Alternatively, the differential equation can be solved using integrating factors, since it is linear and can be written in the form

$$ty' + y = 3t^2 \quad\Rightarrow\quad y' + \frac{1}{t}y = 3t$$

The integrating factor is $e^{\ln t} = t$, and we would get $(ty)' = 3t^2$, and therefore $ty = t^3 + K$, or $y = t^2 + K/t$.

(c) The differential equation $y' = 2y(3-y)$ is autonomous, with $F(y) = 2y(3-y)$. The equilibrium states are therefore given by

$$F(y) = 2y(3 - y) = 0 \quad\Rightarrow\quad y = 0 \quad\text{or}\quad y = 3$$

Hence $y_e = 0$ and $y_e = 3$ are the equilibrium states. To determine their stability, we compute $F'(y_e)$. Since $F'(y) = (6y - 2y^2)' = 6 - 4y$, we get

$$F'(0) = 6 > 0, \quad F'(3) = -6 < 0$$

Therefore, $y_e = 0$ is unstable and $y_e = 3$ is stable by the Stability Theorem. We can also see this from the phase diagram below, where the arrows show the time development of $y = y(t)$ as time passes. None of the equilibrium states are globally asymptotically stable, since an initial value $y_0 < 0$ will give a solution curve that moves away from both equilibrium states as time passes. In fact, since $y' = F(y) = 2y(3 - y) < 0$ for $y < 0$, the solution curve will be decreasing.

Question 3.

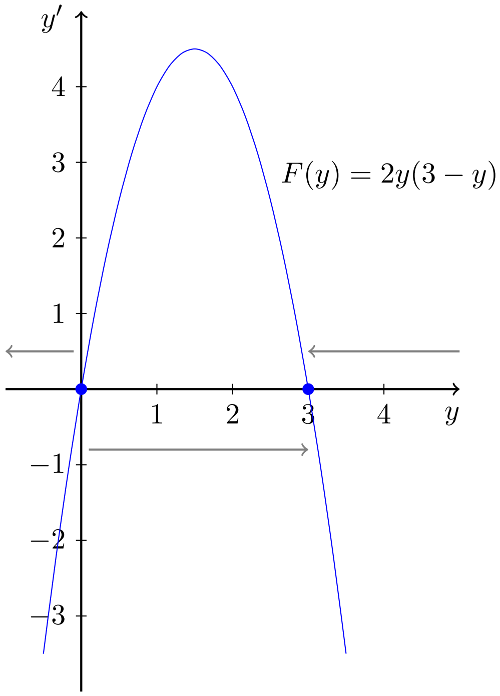
(a) The stationary points of u are given by

$$u'_x = 2x + 4y = 0, \quad u'_y = 4x + 10y - 2z = 0, \quad -2y + 16z = 0$$

This gives $y = 8z$ from the last equation, $x = -2y = -2(8z) = -16z$ from the first equation, and therefore $4(-16z) + 10(8z) - 2z = 0$ from the second equation, or $14z = 0$. Thie implies that $z = 0$, and $(x, y, z) = (0, 0, 0)$ is therefore the unique stationary point of u. The Hessian of u is given by

$$H(u) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & -2 \\ 0 & -2 & 16 \end{pmatrix}$$

and since $D_1 = 2$, $D_2 = 20 - 16 = 4$ and $D_3 = 16(4) + 2(-4) = 56$ (by cofactor expansion along the last row), it follows that $H(u)$ is positive definite and that u is a convex function. Therefore, $u(0, 0, 0) = 1$ is the minimum value of u.



(b) The outer function $f(u) = \ln(u)/u^2$ has derivative

$$f'(u) = \frac{(1/u) \cdot u^2 - \ln(u) \cdot 2u}{u^4} = \frac{1 - 2\ln(u)}{u^3}$$

Therefore, the partial derivatives of f are given by

$$f'_x = \frac{1-2\ln(u)}{u^3} \cdot u'_x = \frac{1-2\ln(u)}{u^3} \cdot (2x + 4y)$$

$$f'_y = \frac{1-2\ln(u)}{u^3} \cdot u'_y = \frac{1-2\ln(u)}{u^3} \cdot (4x + 10y - 2z)$$

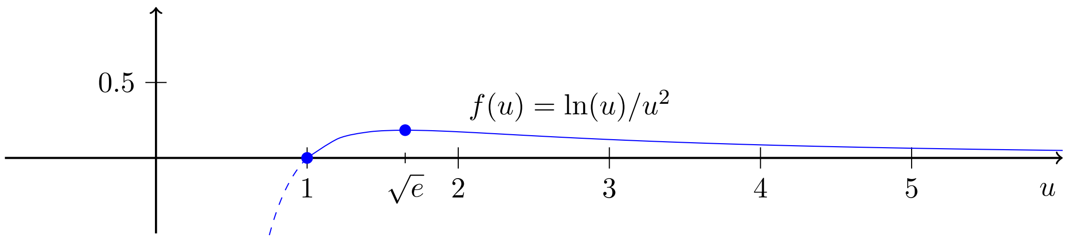
$$f'_z = \frac{1-2\ln(u)}{u^3} \cdot u'_z = \frac{1-2\ln(u)}{u^3} \cdot (-2y + 16z)$$

with $u = 1 + x^2 + 5y^2 + 8z^2 + 4xy - 2yz$.

(c) From (a) we know that the values of the inner function are $u \geq 1$, and from (b) we know that $f'(u) = (1 - 2\ln u)/u^3$ is the derivative of the outer function. This means that $f'(u) = 0$ for $1 - 2\ln(u) = 0$, or $u = e^{1/2} = \sqrt{e}$, that $f(u)$ is increasing for u in $[1, \sqrt{e}]$, and that $f(u)$ is decreasing for u in $[\sqrt{e}, \infty)$. When $u \to \infty$, we have that $f(u) = \ln(u)/u^2 \to 0$. This means that the maximum and minimum values of f are

$$f_{\max} = f(\sqrt{e}) = \frac{1}{2e} \cong 0.184, \quad f_{\min} = f(1) = 0$$

since $f(\sqrt{e}) = 1/(2e)$ and $f(1) = 0$.



QUESTION 4.

(a) The Lagrangian of the Kuhn-Tucker problem is $\mathcal{L} = x^2y^2 - \lambda(x^2 + y^2 + x^2y^2)$. The first order conditions (FOC) are

$$\mathcal{L}'_x = 2xy^2 - \lambda(2x + 2xy^2) = 0$$
$$\mathcal{L}'_y = 2yx^2 - \lambda(2y + 2yx^2) = 0$$

the constraint (C) is given by $x^2 + y^2 + x^2y^2 \le 3$, and the complentary slackness conditions (CSC) are given by

$$\lambda \ge 0 \quad \text{and} \quad \lambda(x^2 + y^2 + x^2y^2 - 3) = 0$$

The Kuhn-Tucker conditions are FOC+C+CSC.

(b) We look at the cases when (i) $x^2 + y^2 + x^2y^2 = 3$ and (ii) $x^2 + y^2 + x^2y^2 < 3$ separately. In each case, we find all points $(x, y; \lambda)$ with $x, y \ne 0$ that satisfies FOC+C+CSC. We start with case (i): We write the FOC's in factorized form:

$$2x(y^2 - \lambda(1 + y^2)) = 0$$
$$2y(x^2 - \lambda(1 + x^2)) = 0$$

This means that $y^2 = \lambda(1 + y^2)$ from the first equation, and that $x^2 = \lambda(1 + x^2)$ from the second equation, since we want to find solutions with $x, y \ne 0$, and therefore

$$\lambda = \frac{x^2}{1 + x^2} = \frac{y^2}{1 + y^2}$$

Multiplication with the common denominator $(1+x^2)(1+y^2) \ne 0$ gives $x^2(1+y^2) = y^2(1+x^2)$, or $x^2 + x^2y^2 = y^2 + x^2y^2$, and this implies that $x^2 = y^2$. When we put this into the constraint, we get

$$x^2 + x^2 + x^2 \cdot x^2 = 3 \quad \Rightarrow \quad x^4 + 2x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = \frac{-2 \pm \sqrt{4 + 12}}{2} = \frac{-2 \pm 4}{2}$$

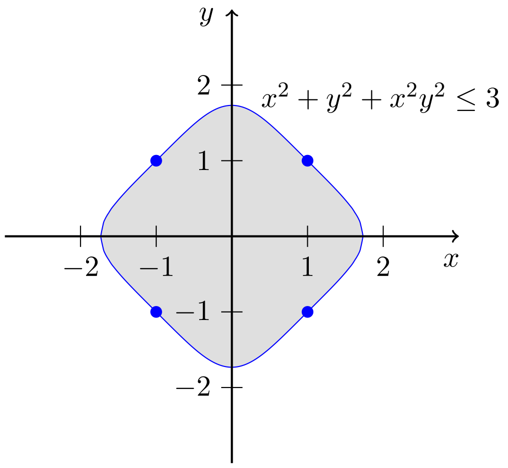
We get that $x^2 = 1$, or $x = \pm 1$, or that $x^2 = -3$, which is not possible. Since $x^2 = 1$, we get $\lambda = 1/2 \ge 0$, and $x^2 = y^2$ means that $y^2 = 1$, or $y = \pm 1$. Hence we get four candidate points in case (i) with $x, y \ne 0$, given by

$$(x, y; \lambda) = (\pm 1, \pm 1; 1/2)$$

All these candidate points have $f = x^2y^2 = 1$. In case (ii), where $x^2 + y^2 + x^2y^2 < 3$ and $\lambda = 0$, the FOC's give $2xy^2 = 0$ and $2x^2y = 0$. This means that $x = 0$ or $y = 0$, and there are no candidate points in case (ii) with $x, y \ne 0$. In conclusion, the points $(x, y; \lambda)$ with $x, y \ne 0$ that satisfied the Kuhn-Tucker conditions FOC+C+CSC, are

$$(x, y; \lambda) = (\pm 1, \pm 1; 1/2)$$

shown in the figure below.



(c) The set of points (x, y) such that $x^2 + y^2 + x^2y^2 = 3$ is bounded. In fact, $x^2, y^2, x^2y^2 \geq 0$ and therefore $x^2, y^2, x^2y^2 \leq 3$, which means that $-\sqrt{3} \leq x, y \leq \sqrt{3}$. By the EVT, the Kuhn-Tucker problem therefore has a maximum. The possible maximum points are the candidate points with $x, y \neq 0$ found in (b), candidate points with $x = 0$ or $y = 0$ that satisfy FOC+C+CSC, and admissible points where NDCQ fails. The candidate points found in (b) have $f = 1$. Therefore, possible candidate points with $x = 0$ or $y = 0$, where $f = 0$, cannot be maximum points. The NDCQ in the case $x^2 + y^2 + x^2y^2 = 3$ is given by

$$\operatorname{rk}\begin{pmatrix} 2x + 2xy^2 & 2y + 2yx^2 \end{pmatrix} = 1$$

and it fails if $2x + 2xy^2 = 0$ and $2y + 2yx^2 = 0$, which gives $2x(1 + y^2) = 0$ and $2y(1 + x^2) = 0$. Since $1 + x^2, 1 + y^2 > 0$, this is the case only at the point $x = y = 0$, and this point does not satisfy $x^2 + y^2 + x^2y^2 = 3$. Since there is no NDCQ condition in case $x^2 + y^2 + x^2y^2 < 3$, there are no admissible points where NDCQ fails. We conclude that $f = 1$ is the maximum value, and that $(x, y; \lambda) = (\pm 1, \pm 1; 1/2)$ are the maximum points. As an alternative method, one may try to use SOC at one of these points, but the corresponding function

$$h(x, y) = \mathcal{L}(x, y; 1/2) = x^2y^2 - \frac{1}{2}\left(x^2 + y^2 + x^2y^2\right)$$

is not concave, and the SOC gives no conclusion in this case.

Question 5.

The system of first order linear differential equations can be written in the matrix form $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 5 & -6 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \mathbf{y}' = \begin{pmatrix} y' \\ z' \end{pmatrix}$$

To solve the system, we find the eigenvalues and eigenvectors of A. The characateristic equation is $\lambda^2 - 3\lambda - 4 = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = 4$. For $\lambda = -1$, the eigenvectors are the solutions of $(A + I)\mathbf{y} = \mathbf{0}$, with

$$A + I = \begin{pmatrix} 6 & -6 \\ 1 & -1 \end{pmatrix}$$

Therefore, $y = z$ with z free. For $\lambda = 4$, the eigenvectors are the solutions of $(A - 4I)\mathbf{y} = \mathbf{0}$, with

$$A - 4I = \begin{pmatrix} 1 & -6 \\ 1 & -6 \end{pmatrix}$$

Therefore, $y = 6z$ with z free. If follows that $E_{-1} = \operatorname{span}(\mathbf{v}_1)$ and $E_4 = \operatorname{span}(\mathbf{v}_2)$ with

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

It follows that the general solution of the system is given by

$$\mathbf{y} = C_1\mathbf{v}_1 \cdot e^{-t} + C_2\mathbf{v}_2 \cdot e^{4t} \quad \Rightarrow \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C_1\, e^{-t} + 6C_2\, e^{4t} \\ C_1\, e^{-t} + C_2\, e^{4t} \end{pmatrix}$$