Solutions	Final exam in GRA 6035 Mathematics
Date	December 1st, 2016 at 0900 - 1200

Question 1.

(a) The determinant of A is

$$\det(A) = \begin{vmatrix} -2 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{vmatrix} = -2(2) - 2(-2) = 0$$

This means that $\text{rk}(A) < 3$, and since at least one of the 2-minors are non-zero, for instance

$$\begin{vmatrix} -2 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0$$

it follows that $\text{rk}(A) = 2$.

(b) The linear system $A\mathbf{x} = \mathbf{0}$ has one free variables since A has rank two, and we compute the solutions using Gaussian elimination:

$$\begin{pmatrix} -2 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This means that z is free, $y = 2z$ and $x = 2z$, and the solutions to the linear system can be written in the form

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ 2z \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = z \cdot \mathbf{v}_1, \quad \text{with } \mathbf{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

That is, the solutions are the vectors in $\text{span}(\mathbf{v}_1)$.

(c) The eigenvalues of A are given by the characteristic equation $\det(A - \lambda I) = 0$, which becomes

$$\begin{vmatrix} -2-\lambda & 2 & 0 \\ -1 & -\lambda & 2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

This gives the equation

$$(-2-\lambda)(-\lambda(2-\lambda) + 2) - 2(-1)(2-\lambda) = 0$$

which gives, after multiplication, that

$$(-2-\lambda)(\lambda^2 - 2\lambda + 2) + 2(2-\lambda) = -\lambda^3 = 0$$

The eigenvalues are therefore $\lambda = 0$ (with multiplicity three).

Question 2.

(a) The difference equation can be written $y_{t+2} - 3y_{t+1} + 2y_t = 0$ and is second order linear. It has characteristic equation $r^2 - 3r + 2 = 0$, with solutions $r = 1$ and $r = 2$. Therefore, the general solution is

$$y_t = C_1 \cdot 1^t + C_2 \cdot 2^t = C_1 + C_2 \cdot 2^t$$

The initial conditions are $y_0 = C_1 + C_2 = 1$ and $y_1 = C_1 + 2C_2 = 2$, which gives $C_2 = 1$ and $C_1 = 0$. The solution is therefore

$$y_t = 2^t$$

(b) The differential equation $y' - y\ln t = y$ is both linear and separable since it can be written $y' = y(\ln t + 1)$. We solve it as a linear differential equation $y' - (\ln t + 1)y = 0$. Since

$$\int \ln t \, \mathrm{d}t = t\ln t - t + C$$

(using integration by parts with $u' = 1$ and $v = \ln t$), it follows that

$$\int -(\ln t + 1)\, \mathrm{d}t = -(t\ln t - t + t) + C = -t\ln t + C$$

and that $e^{-t\ln t}$ is an integrating factor (with $C = 0$). Therefore, the diffential equation can be written $(y\, e^{-t\ln t})' = 0$, which gives

$$y\, e^{-t\ln t} = K \quad \Rightarrow \quad y = K\, e^{t\ln t}$$

We could also solve it as a separable differential equation

$$y' = y(\ln t + 1) \quad \Rightarrow \quad \frac{1}{y} \cdot y' = \ln t + 1 \quad \Rightarrow \quad \int \frac{1}{y}\, \mathrm{d}y = \int \ln t + 1 \, \mathrm{d}t$$

The integral on the right hand side is computed as shown above. This gives

$$\ln|y| = t\ln t + \mathcal{C} \quad \Rightarrow \quad |y| = e^{t\ln t + \mathcal{C}} \quad \Rightarrow \quad y = Ke^{t\ln t}$$

with $K = \pm e^{\mathcal{C}}$.

(c) The differential equation $ye^{yt} + te^{yt}y' = 1$ is not separable or linear, and we try to solve it as an exact differential equation. We write it in the form $(ye^{yt} - 1) + (te^{yt})y' = 0$, and try to find a function $h = h(y, t)$ such that

$$h'_t = ye^{yt} - 1, \quad h'_y = te^{yt}$$

From the first equation, it follows that $h = e^{yt} - t + C(y)$, since the derivative $(e^u)'_t = e^u \cdot y$ when $u = yt$ and $u'_t = y$. We check the second equation, and compute

$$h'_y = (e^{yt} - t + C(y))'_y = te^{yt} + C'(y)$$

Therefore $h = e^{yt} - t + C(y)$ is a solution to both equations if $C'(y) = 0$, and the simplest solution to this is $C(y) = 0$. We therefore have that

$$h(y, t) = e^{yt} - t = K \quad \Rightarrow \quad e^{yt} = t + K$$

The initial condition $y(1) = \ln 2$ gives $2 = 1 + K$, or $K = 1$. Hence the solution is

$$yt = \ln(t + K) = \ln(t + 1) \quad \Rightarrow \quad y = \frac{\ln(t+1)}{t}$$

Question 3.

(a) To find out if $f(x, y, z) = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2 + 1$ is convex, we compute its first order partial derivatives

$$f'_x = 10x - 8y - 4z, \quad f'_y = -8x + 10y - 4z, \quad f'_z = -4x - 4y + 16z$$

and its Hessian matrix

$$H(f) = \begin{pmatrix} 10 & -8 & -4 \\ -8 & 10 & -4 \\ -4 & -4 & 16 \end{pmatrix}$$

The leading principal minors are $D_1 = 10$, $D_2 = 36$ and $D_3 = 16 \cdot 36 + 4(-72) - 4(72) = 0$. We have used cofactor expansion along the last row to compute D_3. We see that the Hessian $H(f)$ may be positive semidefinite, and we must check if all principal minors $\Delta_i \geq 0$ to verify this. We compute that $\Delta_1 = 10, 10, 16 > 0$, $\Delta_2 = 36, 144, 144 > 0$ and $\Delta_3 = 0$. Hence $H(f)$ is positive semidefinite, and f is convex.

(b) The stationary points of f are the solutions of the first order conditions, given by

$$f'_x = 10x - 8y - 4z = 0, \quad f'_y = -8x + 10y - 4z = 0, \quad f'_z = -4x - 4y + 16z = 0$$

This is a linear system, and we solve it using Gassian elimination:

$$\begin{pmatrix} 10 & -8 & -4 \\ -8 & 10 & -4 \\ -4 & -4 & 16 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & -4 \\ 0 & -18 & 36 \\ 0 & 18 & -36 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We have divided the last row by -4, and moved it to the top row, to simplify computations. We see that z is a free variable, that $y = 2z$ and that $x = -y + 4z = -2z + 4z = 2z$. Therefore, there are infinitely many stationary points, given by $(x, y, z) = (2z, 2z, z)$.

(c) The minimal value of f is $f = 1$ since f is convex and $(0, 0, 0)$ is one of its stationary points, with $f(0, 0, 0) = 1$. We have that $g(x, y, y) = w \ln(w)$ with $w = f(x, y, z) \geq 1$. We can therefore think of $g(x, y, z)$ as the composite function $h(f(x, y, z))$, where $h(w) = w \ln(w)$ is a function defined for $w \geq 1$. Since $h'(w) = 1 \ln(w) + w(1/w) = \ln(w) + 1 > 0$, it follows that h is a strictly increasing function, and the value of $g(x, y, z)$ is minimal when $f(x, y, z)$ is minimal. Therefore, the minimum value of $g(x, y, z)$ is $h(1) = 1 \cdot \ln(1) = 0$, and this value is attained when $f(x, y, z) = 1$; that is, for all the stationary points $(x, y, z) = (2z, 2z, z)$.

Question 4.

(a) The Lagrangian is $\mathcal{L} = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2 + 1 - \lambda(x + y - 4z)$. The first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}'_x &= 10x - 8y - 4z - \lambda = 0 \\ \mathcal{L}'_y &= -8x + 10y - 4z - \lambda = 0 \\ \mathcal{L}'_y &= -4x - 4y + 16z + 4\lambda = 0 \end{aligned}$$

and the constraint (C) is given by $x + y - 4z = 8$. The Lagrange conditions therefore give a 4×4 linear system $A \cdot \mathbf{x} = \mathbf{b}$, with augmented matrix $(A|\mathbf{b})$, given by

$$\begin{pmatrix} 1 & 1 & -4 & 0 \\ 10 & -8 & -4 & -1 \\ -8 & 10 & -4 & -1 \\ -4 & -4 & 16 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ \lambda \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (A|\mathbf{b}) = \left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & 8 \\ 10 & -8 & -4 & -1 & 0 \\ -8 & 10 & -4 & -1 & 0 \\ -4 & -4 & 16 & 4 & 0 \end{array}\right)$$

when the columns correspond to the variables x, y, z, λ and we write the constraint (C) first and then the first order conditions (FOC).

(b) We use Gaussian elimination to solve the Lagrange conditions, given by the linear system given in a):

$$\left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & 8 \\ 10 & -8 & -4 & -1 & 0 \\ -8 & 10 & -4 & -1 & 0 \\ -4 & -4 & 16 & 4 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & 8 \\ 0 & -18 & 36 & -1 & -80 \\ 0 & 0 & 0 & -2 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This shows that there are infinitely many solutions to the Lagrange conditions, with z free, with $\lambda = 8$, with $-18y = -36z + 8 - 80 = -36z - 72$ which gives $y = 2z + 4$, and with $x = -y + 4z + 8 = 4z + 8 - (2z + 4) = 2z + 4$. In other words, the solutions are

$$(x, y, z; \lambda) = (2z + 4, 2z + 4, z; 8)$$

for any value of z. We choose one of the these points, for example the point $(4, 4, 0; 8)$ with $z = 0$, and use the SOC: The Lagrangian

$$\mathcal{L}(x, y, z; 8) = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2 - 8(x + y - 4z)$$

has the same Hessian matrix as f in Question 3a). It follows that $\mathcal{L}$ is a convex function, and therefore $(4, 4, 0)$ is a minimum point with minimum value $f(4, 4, 0) = 33$. Any of the other solutions $(x, y, z) = (2z + 4, 2z + 4, z)$ of the Lagrange conditions is also a minimum point with $f(2z + 4, 2z + 4, z) = 33$, since it gives the same Lagrangian.

(c) We consider the Lagrange problem $\min f(x,y,z)$ subject to $x+y-4z=a$. Its Lagrangian is

$$\mathcal{L} = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2 + 1 - \lambda(x+y-4z-a)$$

and we see that there is a solution $(x^*(a), y^*(a), z^*(a); \lambda^*(a))$ of the Lagrange conditions for each value of a. In fact, we find such solutions by replacing the linear system in b) with the linear system

$$\left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & a \\ 10 & -8 & -4 & -1 & 0 \\ -8 & 10 & -4 & -1 & 0 \\ -4 & -4 & 16 & 4 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & a \\ 0 & -18 & 36 & -1 & -10a \\ 0 & 0 & 0 & -2 & -2a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and see that $(x^*(a), y^*(a), z^*(a); \lambda^*(a)) = (2z + a/2, 2z + a/2, z; a)$ are solutions for all values of z. By the SOC, these solutions are minima for all values of z since the Lagrangian

$$\mathcal{L} = 5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2 + 1 - a(x+y-4z-a)$$

is a convex function (it has the same Hessian as the Lagrangian in b). This implies that we can use the envelope theorem

$$\frac{\mathrm{d}f^*(a)}{\mathrm{d}a} = \mathcal{L}'_a(x^*(a), y^*(a), z^*(a); \lambda^*(a))$$

The right hand side is equal to $\lambda^*(a) = a$ since $\mathcal{L}'_a = \lambda$, and it follows that $df^*(a)/da = 8$ at $a = 8$. We estimate the new minimal value at $a = 7.92$ as

$$f^*(7.92) \cong f^*(8) + 8 \cdot \Delta a = 33 + 8 \cdot (-0.08) = 33 - 0.64 = 32.36$$

One finds that the exact value is $f^*(a) = f(x^*(a), y^*(a), z^*(a); \lambda^*(a)) = 1 + a^2/2$, which gives $f^*(7.92) = 32.3632$ is the exact solution to the new Lagrange problem.

Question 5.

To compute the rank, we first find the determinant, and use cofactor expansion along the first row:

$$|A| = \begin{vmatrix} -\alpha_2 & \alpha_1 & 0 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 \end{vmatrix} = -\alpha_2(\alpha_1\alpha_3) - \alpha_1(-\alpha_2\alpha_3) = 0$$

Since $\det(A) = 0$, this means that $\operatorname{rk} A \le 2$, and we see that $\operatorname{rk} A = 0$ if $(\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$. When $(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0)$, we compute 2-minors to see if $\operatorname{rk} A = 2$. Among the 2-minors, we look at

$$M_{12,23} = \begin{vmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{vmatrix} = \alpha_1^2, \qquad M_{13,13} = \begin{vmatrix} -\alpha_2 & 0 \\ 0 & \alpha_2 \end{vmatrix} = -\alpha_2^2, \qquad M_{23,12} = \begin{vmatrix} -\alpha_3 & 0 \\ 0 & -\alpha_3 \end{vmatrix} = \alpha_3^2$$

and notice that if $(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0)$, then at least one of these minors are non-zero, since at least one $\alpha_i \neq 0$. This means that $\operatorname{rk} A = 2$ when $(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0)$. We get

$$\operatorname{rk} A = \begin{cases} 2, & (\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0) \\ 0, & (\alpha_1, \alpha_2, \alpha_3) = (0,0,0) \end{cases}$$