Solutions	Final exam in GRA 6035 Mathematics
Date	March 4th, 2016 at 1500 - 1800

Question 1.

(a) The rank of A is two since it has an echelon form

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear system $A\mathbf{x} = \mathbf{0}$ has free variables z, w and the solution is given by the equations $x + z = 0$ and $y + w = 0$, which gives $x = -z$ and $y = -w$, or

$$\mathbf{x} = \begin{pmatrix} -z \\ -w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

(b) Since A is symmetric, it is diagonalizable. The eigenvalues are given by the characteristic equation $\det(A - \lambda I) = 0$, which becomes

$$\begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

Cofactor expansion along the first column gives

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & 0 \\ 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

and cofactor expansion along the second column in each of the new minors give

$$(1-\lambda)^2((1-\lambda)^2 - 1) - 1((1-\lambda)^2 - 1) = 0$$

Since $(1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$, this gives

$$((1-\lambda)^2 - 1)(\lambda^2 - 2\lambda) = (\lambda^2 - 2\lambda)^2 = \lambda^2(\lambda - 2)^2 = 0$$

The eigenvalues are therefore $\lambda = 0$ (with multiplicity two) and $\lambda = 2$ (with multiplicity two).

(c) The quadratic form is $Q(x, y, z, w) = x^2 + 2xz + y^2 + 2yw + z^2 + w^2$. Since $\lambda = 0$ and $\lambda = 2 > 0$ are the eigenvalues of the symmetric matrix A, it follows that A is positive semidefinite (but not positive definite). Alternatively, we may write $Q = (x+z)^2 + (y+w)^2 \geq 0$ for all x, y, z, w, or we may compute the leading principal minors

$$D_1 = 1,\ D_2 = 1,\ D_3 = 0,\ D_4 = 0$$

and all principal minors (since $D_3 = 0$), given by

$$\Delta_1 = 1, 1, 1, 1 \quad \Delta_2 = 1, 0, 1, 1, 0, 1 \quad \Delta_3 = 0 \quad \Delta_4 = 0$$

where all $\Delta_3, \Delta_4 = 0$ since A has rank two. The conclusion is that A is positive semidefinite since all principal minors $\Delta_i \geq 0$.

(d) We know that $\mathbf{x}_{t+1} = T\mathbf{x}_t$ is a regular Markov chain, since all entries in T are positive, so the equilibrium state $\mathbf{x}$ is the unique eigenvalue of T with eigenvalue $\lambda = 1$ that is a state vector. We compute the eigenvectors with eigenvalue $\lambda = 1$:

$$\begin{pmatrix} -0.25 & 0.25 & 0.10 \\ 0.10 & -0.40 & 0.05 \\ 0.15 & 0.15 & -0.15 \end{pmatrix} \to \begin{pmatrix} -5 & 5 & 2 \\ 2 & -8 & 1 \\ 3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -0.40 \\ 0 & 6 & -1.80 \\ 0 & -6 & 1.80 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & -0.40 \\ 0 & 1 & -0.30 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that z is free, that $y = 0.30z$ and that $x = y + 0.40z = 0.70z$. The unique eigenvector that is a state vector is given by

$$x + y + z = 1 \quad \Rightarrow \quad 0.70z + 0.30z + z = 2z = 1$$

which gives $z = 1/2$. This gives equilibrium state $\mathbf{x}$ with

$$x = 0.35 \quad y = 0.15 \quad z = 0.50$$

Question 2.

(a) The differential equation $y'' - 16y = e^{-t}$ is second order linear, and has general solution $y = y_h + y_p$. The homogeneous solution is

$$y_h = C_1 e^{4t} + C_2 e^{-4t}$$

since the characteristic equation $r^2 - 16 = 0$ has solutions $r = 4$ and $r = -4$. To find the particular solution, we guess a solution of the form $y = Ae^{-t}$, since $f(t) = e^{-t}$, $f'(t) = -e^{-t}$ and $f''(t) = e^{-t}$. We compute $y' = -Ae^{-t}$, $y'' = Ae^{-t}$, which gives

$$Ae^{-t}(1 - 16) = e^{-t} \quad \Rightarrow \quad -15Ae^{-t} = e^{-t}$$

We see that $A = -1/15$ is a solution, so $y_P = -e^{-t}/15$ and the general solution is

$$y = C_1 e^{4t} + C_2 e^{-4t} - \frac{1}{15} e^{-t}$$

(b) The differential equation $(3t^2y + 2ty^2 + t^3) + (t^3 + 2yt^2)y' = 0$ can be written in the form $p + qy' = 0$ with

$$p = 3t^2y + 2ty^2 + t^3, \quad q = t^3 + 2yt^2$$

We attempt to find a function $h = h(y, t)$ such that $h'_t = p$ and $h'_y = q$. From the first equation, we see that $h = t^3y + t^2y^2 + t^4/4 + \phi(y)$, since $(^3y + t^2y^2 + t^4/4)'_t = 3t^2y + 2ty^2 + t^3 = p$. Using this expression for h, the second condition becomes

$$h'_y = t^3 + 2t^2y + \phi'(y) = t^3 + 2yt^2$$

which is satisfied if $\phi'(y) = 0$, and one solution is $\phi(y) = 0$. This implies that differential equation $p + qy' = 0$ is exact and that $h = t^3y + t^2y^2 + t^4/4$ satisfies $h'_t = p$ and $h'_y = q$. The solution of the differential equation is therefore

$$t^3y + t^2y^2 + t^4/4 = \mathcal{C}$$

To find an explicit solution, we solve for y using the abc-formula:

$$y = \frac{-t^3 \pm \sqrt{t^6 - 4t^2(t^4/4 - C)}}{2t^2} = \frac{-t^3 \pm \sqrt{4Ct^2}}{2t^2} = -\frac{t}{2} \pm \frac{\sqrt{C}}{t}$$

(c) The differential equation $y' = yt/\ln(y)$ is separable, and it can be written in the form

$$\frac{\ln y}{y} y' = t \quad \Leftrightarrow \quad \int \frac{\ln y}{y}\, dy = \int t\, dt$$

The substitution $u = \ln y$, with $\mathrm{d}u = (1/y)\mathrm{d}y$ gives that

$$\int \frac{\ln y}{y}\, dy = \int u\, \mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln y)^2 + C$$

Therefore, integration on both sides of the differential ewquation gives

$$\frac{1}{2}(\ln y)^2 = \frac{1}{2}t^2 + C \quad \Leftrightarrow \quad (\ln y)^2 = t^2 + 2C$$

The initial condition $y(0) = e$ gives $1^2 = 0 + 2C$, or $C = 1/2$, and we find

$$\ln y = \sqrt{t^2 + 1} \quad \Leftrightarrow \quad y = e^{\sqrt{t^2+1}}$$

Question 3.

(a) The partial derivatives of $f(x, y, z) = e^{1-u}$, with $u = x^2 + 2xy + 3y^2 + 2yz + z^2$, are given by

$$\begin{aligned} f'_x &= e^{1-u} \cdot (-u'_x) = e^{1-u}(-2x - 2y) \\ f'_y &= e^{1-u} \cdot (-u'_y) = e^{1-u}(-2x - 6y - 2z) \\ f'_z &= e^{1-u} \cdot (-u'_z) = e^{1-u}(-2y - 2z) \end{aligned}$$

Since $e^{1-u} > 0$, the stationary points are the solutions to the linear equations

$$-2x - 2y = 0, \quad -2x - 6y - 2z = 0, \quad -2y - 2z = 0$$

The last equation gives $y = -z$, and the first that $x = -y = z$, and this implies that $-2z + 6z - 2z = 0$ by the middle equation. This gives $z = 0$, and therefore $x = y = z = 0$. The function f has one stationary point $(x, y, z) = (0, 0, 0)$.

(b) To classify $(0, 0, 0)$, we compute the Hessian $H(f)(0, 0, 0)$ at this point. We find that

$$f''_{xx} = e^{1-u}(-u''_{xx}) + e^{1-u}(-u'_x) \cdot (-u'_x) = e^{1-u}((u'_x)^2 - u''_{xx}) = e^{1-u}(4(x + y)^2 - 2)$$

The other second order partial derivatives can be computed in a similar way. In f''_{ab}, where a, b are two of the variables x, y, z, we see that only the non-zero term after we substitute $(x, y, z) = (0, 0, 0)$ is

$$f''_{ab} = e^{1-u} \cdot (-u''_{ab}) = -e \cdot u''_{ab}$$

We therefore find the Hesssian

$$H(f)(0, 0, 0) = -e \cdot H(u) = -e \begin{pmatrix} 2 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Since $D_1 = -2e < 0$, $D_2 = (12 - 4)e^2 = 8e^2 > 0$ and $D_3 = -(16 - 8)e^3 = -8e^3 < 0$, it follows that $H(f)(0, 0, 0)$ is negative definite and that the stationary point $(0, 0, 0)$ is a local maximum for f.

(c) We may write $f(x, y, z) = e^{1-u(x,y,z)} = e^w$ with $w = 1 - u$, and we known that e^w is an increasing function in w since $(e^w)'_w = e^w > 0$. This means that

$$w_1 < w_2 \quad \Rightarrow \quad e^{w_1} \leq e^{w_2}$$

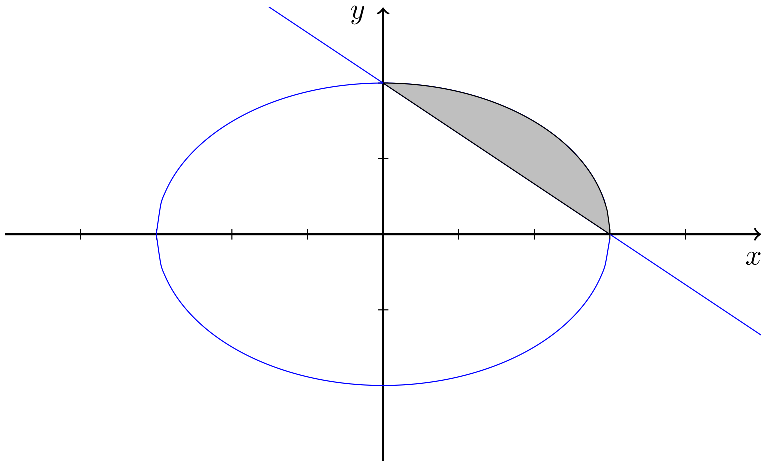
and f has a global maximum if $w = 1 - u$ has a global maximum. We have that

$$H(1 - u) = H(-u) = \begin{pmatrix} -2 & -2 & 0 \\ -2 & -6 & -2 \\ 0 & -2 & -2 \end{pmatrix}$$

is negative definite. In fact, it is equal to $H(f)(0, 0, 0)/e$ since $H(f)(0, 0, 0) = -eH(u)$, and $H(f)(0, 0, 0)$ is negative definite. But we may also show that $H(1 - u)$ is negative definite directly, since $H(1 - u) = H(-u)$ has leading principal minors $D_1 = -2$, $D_2 = 8$ and $D_3 = 8$. It follows that $1 - u$ is a concave function. It has stationary point $(x, y, z) = (0, 0, 0)$ by the calculation in b). This point is therefore a global maximum for $w = 1 - u$, with maximum value $w = 1 - u(0, 0, 0) = 1 - 0 = 1$. Finally, it follows that $f(x, y, z) = e^w = e^{1-u}$ has a global maximum at $(x, y, z) = (0, 0, 0)$ with maximum value $f = e^1 = e$.

Question 4.

(a) The boundary of the set of R of admissible points is given by the equations $4x^2 + 9y^2 = 36$ and $2x + 3y = 6$. The graph of the linear equation is the straight line $y = 2 - x/3$, and the quadratic equation is the equation of an ellipse with center in the origin and half-axis 3 (in the x-direction) and 2 (in the y-direction). The region R of admissible points is the region bounded by these two curves, above the straight line and inside the ellipse, and is shown in the figure below. It is clear from the figure that R is bounded, since $0 \leq x \leq 3$ and $0 \leq y \leq 2$.



(b) We write the Kuhn-Tucker problem in standard form as

$$\max\ f(x,y) = xy \text{ subject to } \begin{cases} 4x^2 + 9y^2 \le 36 \\ -(2x+3y) \le -6 \end{cases}$$

It has Lagrangian $\mathcal{L} = xy - \lambda_1(4x^2+9y^2) + \lambda_2(2x+3y)$. The first order conditions (FOC) are

$$\mathcal{L}'_x = y - \lambda_1 \cdot 8x + 2\lambda_2 = 0$$
$$\mathcal{L}'_y = x - \lambda_1 \cdot 18y + 3\lambda_2 = 0$$

the constraints (C) are given by $4x^2 + 9y^2 \le 36$ and $2x + 3y \ge 6$, and the complementary slackness conditions (CSC) are given by

$$\lambda_1, \lambda_2 \ge 0 \quad \text{and } \lambda_1(4x^2 + 9y^2 - 36) = \lambda_2(2x + 3y - 6) = 0$$

The FOC's is a linear system with variables x, y and parameters λ_1, λ_2, with augmented matrix

$$\left(\begin{array}{cc|c} -8\lambda_1 & 1 & -2\lambda_2 \\ 1 & -18\lambda_1 & -3\lambda_2 \end{array}\right)$$

When $\lambda_1 = 1/12$, it becomes

$$\left(\begin{array}{cc|c} -2/3 & 1 & -2\lambda_2 \\ 1 & -3/2 & -3\lambda_2 \end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{cc|c} 1 & -3/2 & -3\lambda_2 \\ 0 & 0 & -4\lambda_2 \end{array}\right)$$

This linear system is only consistent if $\lambda_2 = 0$, and in this case we have $x - 3y/2 = 0$, or $x = 3y/2$. Since $\lambda_1 = 1/12 > 0$, the CSC give

$$4x^2 + 9y^2 = 4(3y/2)^2 + 9y^2 = 18y^2 = 36$$

This means that $y^2 = 2$ and $y = \pm\sqrt{2}$ and $x = \pm 3\sqrt{2}/2$. We find the two solutions

$$(x, y; \lambda_1, \lambda_2) = (3\sqrt{2}/2, \sqrt{2}; 1/12, 0),\ (-3\sqrt{2}/2, -\sqrt{2}; 1/12, 0)$$

with $\lambda_1 = 1/12$. Finally, we check the second constraint $2x + 3y \ge 6$, and see that only the first point satisfy it. Therefore, the Kuhn-Tucker conditions have a unique solution with $\lambda_1 = 1/12$:

$$(x, y; \lambda_1, \lambda_2) = (3\sqrt{2}/2, \sqrt{2}; 1/12, 0)$$

(c) We attempt to use the Second Order Condition (SOC) at the point found in b). We get the Lagrangian

$$L = xy - \frac{1}{12}\left(4x^2 + 9y^2\right)$$

since $\lambda_1 = 1/12$ and $\lambda_2 = 0$. The Hessian becomes

$$H(L) = \begin{pmatrix} -8/12 & 1 \\ 1 & -18/12 \end{pmatrix} = \begin{pmatrix} -2/3 & 1 \\ 1 & -3/2 \end{pmatrix}$$

We have leading principal minors $D_1 = -2/3 < 0$ and $D_2 = 0$, and the other first order principal minor is $\Delta_1 = -3/2 < 0$. It follows that $H(L)$ is negative semidefinite (at all points) and that L therefore is a concave function. From the (SOC) it follows that $(x, y) = (3\sqrt{2}/2, \sqrt{2})$ is a global maximum point, with maximum value $f = 3\sqrt{2}/2 \cdot \sqrt{2} = 3$.