Exam	Final exam in GRA 6035 Mathematics
Date	December 11th, 2015 at 0900 - 1200

All subquestions have the same weight and give maximal score 6p each. Answers to the first 12 subquestions give a maximal score of 72p (100%). Question 4 (c) can be skipped, but gives 6p extra credit if answered correctly.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A and find all the solutions of the linear system $A \cdot \mathbf{x} = \mathbf{0}$.
(b) **(6p)** Is A diagonalizable? Find all eigenvalues of A and their multiplicities.
(c) **(6p)** Compute the eigenvalues of $B = A^2$. Is B diagonalizable?

We consider a Markov chain given by $\mathbf{x}_{t+1} = T\mathbf{x}_t$, where the transition matrix T is given by

$$T = \begin{pmatrix} 0.55 & 0.10 & 0.15 \\ 0.10 & 0.80 & 0.05 \\ 0.35 & 0.10 & 0.80 \end{pmatrix}$$

and the inital state is $\mathbf{x}_0$.

(d) **(6p)** Find the equilibrium state $\mathbf{x} = \lim_{t\to\infty} T^t\, \mathbf{x}_0$

Question 2.

Solve the differential equations:

(a) **(6p)** $y'' - 4y' - 12y = 15e^t$
(b) **(6p)** $y' = 3\sqrt{t} \cdot e^{-2y}$
(c) **(6p)** $4yt + 4t^3 + 2t + (2y - 1 + 2t^2)y' = 0, \quad y(1) = 0$

Question 3.

We consider the function given by $f(x, y, z) = \ln(u + 1)$, where $u = u(x, y, z)$ is the quadratic form

$$u(x, y, z) = 2x^2 + 2xy + 3y^2 - 2xz + z^2$$

(a) **(6p)** Determine the definiteness of u. Are there any points (x, y, z) where f is not defined?
(b) **(6p)** Compute the partial derivatives of f, and find all stationary points of f.
(c) **(6p)** Find the minimum value of f, if it exists. Is f convex?

Question 4.

We consider the following Kuhn-Tucker problem:

$$\min\ f(x,y) = x^2 + y^2 \text{ subject to } xy \geq 4$$

(a) **(6p)** Sketch the set of admissible points. Is it bounded? Explain why there is a minimum. You may use that $\sqrt{x^2+y^2}$ is the distance from the origin $(0,0)$ to a general point (x,y).
(b) **(6p)** Find the minimum value using the Kuhn-Tucker conditions.

We consider the following Kuhn-Tucker problem:

$$\min\ f(x,y,z,w) = x^2 + y^2 + z^2 + w^2 \text{ subject to } \begin{cases} xz \geq 9 \\ yw \geq 25 \end{cases}$$

(c) **Extra credits (6p)** Solve the Kuhn-Tucker problem and find the corresponding minimum value.