Solutions	Final exam in GRA 6035 Mathematics
Date	May 15th, 2015 at 0900 - 1200

Question 1.

(a) The determinant of A is $\det(A) = s^3$, since it is an upper triangular matrix (the determinant is the product of the diagonal entries). Hence, $\operatorname{rk} A = 3$ when $s \neq 0$. When $s = 0$, we have that

$$\operatorname{rk} A = \operatorname{rk} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

since A is an echelon form with no pivot positions. Hence we have

$$\det A = s^3, \quad \operatorname{rk} A = \begin{cases} 3, & s \neq 0 \\ 0, & s = 0 \end{cases}$$

(b) When $s = 1$, the eigenvalues of A are given by the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

Therefore, the only eigenvalues is $\lambda = 1$ (with multiplicity three). The eigenvectors of A with $\lambda = 1$ are given by the linear system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

The coefficient matrix is already an echelon form, so x is a free variable and $y + z = 0$ and $z = 0$, which gives $y = z = 0$. Hence the eigenvectors are given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) For any s, the eigenvalues of A are given by the characteristic equation

$$\begin{vmatrix} s-\lambda & s & s \\ 0 & s-\lambda & s \\ 0 & 0 & s-\lambda \end{vmatrix} = (s-\lambda)^3 = 0$$

Therefore, the only eigenvalues is $\lambda = s$ (with multiplicity three). The eigenvectors of A with $\lambda = s$ are given by the linear system

$$\begin{pmatrix} 0 & s & s \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

If $s \neq 0$, then there is only one degree of freedom (x is free), and therefore A is not diagonalizable. If $s = 0$, then there are three degrees of freedom, and A is diagonalizable (the eigenvalue has multiplicity three). Therefore A is diagonalizable if and only if $s = 0$.

(d) Since T is the transition matrix in a Markov chain, we know that $\lambda = 1$ is an eigenvalue of T. We compute the eigenvectors for $\lambda = 1$ by considering the linear system

$$\begin{pmatrix} -0.30 & 0.30 & 0.50 \\ 0.20 & -0.50 & 0.20 \\ 0.10 & 0.20 & -0.70 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

We compute an echelon form of the coeffficient matrix:

$$\begin{pmatrix} -0.30 & 0.30 & 0.50 \\ 0.20 & -0.50 & 0.20 \\ 0.10 & 0.20 & -0.70 \end{pmatrix} \to \begin{pmatrix} -3 & 3 & 5 \\ 2 & -5 & 2 \\ 1 & 2 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -7 \\ 0 & 9 & -16 \\ 0 & -9 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -7 \\ 0 & 9 & -16 \\ 0 & 0 & 0 \end{pmatrix}$$

There is one degree of freedom (z is free). The equation $9y - 16z = 0$ gives $y = 16z/9$, and the equation $x + 2y - 7z = 0$ gives $x = -2y + 7z = -2(16z/9) + 7z = 31z/9$. The eigenvectors for $\lambda = 1$ are therefore

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{z}{9} \cdot \begin{pmatrix} 31 \\ 16 \\ 9 \end{pmatrix}$$

We know that the long term equilibrium of the Markov chain is the unique eigenvector for $\lambda = 1$ with $x + y + z = 1$ (a market share vector). The equation $z/9(31 + 16 + 9) = 1$ gives $56z/9 = 1$, or $z/9 = 1/56$. Therefore the long term market shares of company A,B,C are given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{56} \begin{pmatrix} 31 \\ 16 \\ 9 \end{pmatrix} = \begin{pmatrix} 31/56 \\ 16/56 \\ 9/56 \end{pmatrix}$$

or $x \approx 55.4\%$ for company A, $y \approx 28.6\%$ for company B, and $z \approx 16.1\%$ for company C.

Question 2.

(a) The equation $y'' - 3y' - 10y = t$ is second order linear, and has solution $y = y_h + y_p$. The characteristic equation is $r^2 - 3r - 10 = 0$, with solutions $r = 5$ and $r = -2$, so the homogeneous solution is $y_y = C_1 e^{5t} + C_2 e^{-2t}$. We guess a particular solution $y_p = At + B$ and put this into the equation $y'' - 3y' - 10y = t$:

$$0 - 3A - 10(At + B) = t \quad \Leftrightarrow \quad -10At + (-10B - 3A) = t$$

This gives $-10A = 1$ and $-10B - 3A = 0$, or $A = -1/10$ and $B = 3/100$. The general solution is therefore

$$y = y_h + y_p = C_1 e^{5t} + C_2 e^{-2t} - \frac{1}{10}t + \frac{3}{100}$$

(b) The equation $t^2 y' + ty = \ln t$ is linear, and can be written as $y' + (1/t)y = (\ln t)/t^2$. It has integrating factor $e^{\ln t} = t$, and after multiplication with the integrating factor, we get

$$(yt)' = \frac{\ln t}{t} \quad \Rightarrow \quad yt = \int \frac{\ln t}{t} dt = \int t^{-1} \ln t \, dt$$

We can solve the integral using the substitution $u = \ln t$, which gives

$$\int t^{-1} \ln t \, dt = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C$$

(or alternatively, we can use integration by parts to solve the integral). We get that

$$yt = \int t^{-1} \ln t \, dt = \frac{1}{2}(\ln t)^2 + C \quad \Rightarrow \quad y = \frac{(\ln t)^2 + 2C}{2t}$$

(c) We rewrite the equation $(\ln t - 6ty)y' = 3y^2 - y/t$ as $(\ln t - 6ty)y' + (y/t - 3y^2) = 0$, and try to solve it as an exact equation. We try to find a function $h = h(y, t)$ such that

$$h'_y = \ln t - 6ty, \quad h'_t = (y/t - 3y^2)$$

The first equation gives $h = y \ln t - 3ty^2 + C(t)$, and when we put the derivative h'_t into the second equation, we get

$$\frac{y}{t} - 3y^2 + C'(t) = \frac{y}{t} - 3y^2$$

Therefore, we see that $C(t) = 0$ gives a solution, and the equation is exact. The general solution of the differential equation is

$$y \ln t - 3ty^2 = K$$

We see that this is a quadratic equation $-3ty^2 + \ln(t)y - K = 0$ in y, and solve it for y to find an explicit solution:

$$y = \frac{-\ln t \pm \sqrt{(\ln t)^2 - 12Kt}}{-6t} = \frac{\ln t}{6t} \pm \frac{\sqrt{(\ln t)^2 - 12Kt}}{6t}$$

Question 3.

(a) The partial derivatives of f are given by

$$f'_x = \frac{2x}{x^2+y^2+1} - 2x, \quad f'_y = \frac{2y}{x^2+y^2+1} - 2y$$

and the stationary point are given by the equations $f'_x = f'_y = 0$, which gives

$$\frac{2x - 2x(x^2+y^2+1)}{x^2+y^2+1} = \frac{-2x(x^2+y^2)}{x^2+y^2+1} = 0, \quad \frac{2y - 2y(x^2+y^2+1)}{x^2+y^2+1} = \frac{-2y(x^2+y^2)}{x^2+y^2+1} = 0$$

or $-2x(x^2+y^2) = -2y(x^2+y^2) = 0$. Either $x^2+y^2 = 0$, or $-2x = -2y = 0$. In either case, the only solution is $x = 0, y = 0$. Hence $(0,0)$ is the unique stationary point of f.

(b) The second order partial derivative f''_{xx} is given by

$$f''_{xx} = \frac{2(x^2+y^2+1) - 2x(2x)}{(x^2+y^2+1)^2} - 2 = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} - 2$$

so $f''_{xx}(1,0) = -2$. We compute f''_{xy} and f''_{yy} in the same way, and get

$$f''_{xy} = \frac{-4xy}{(x^2+y^2+1)^2}, \quad f''_{yy} = \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} - 2$$

This gives $f''_{xy}(1,0) = 0$ and $f''_{yy}(1,0) = -1$. It follows that the Hessian matrix at $(1,0)$ is

$$H(f)(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

This matrix is negative definite since $D_1 = -2 < 0$ and $D_2 = 2 > 0$.

(c) We know that f is concave if and only if the Hessian matrix $H(f)(x,y)$ is negative semidefinite for all points (x,y). It is possible that the Hessian matrix is negative definite at a particular point, but indefinite or positive definite at other points. For instance, $g(x,y) = x^3 + y^3$ has Hessian matrix

$$H(g) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

which is negative definite when $x, y \le 0$, positive definite when $x, y \ge 0$, and indefinite when x and y have opposite signs. Hence, based solely on the fact that $H(f)(x,y)$ is negative definite for a particular point $(x,y) = (1,0)$, we cannot conclude that f is concave.

Question 4.

(a) To write the Kuhn-Tucker problem in standard form, we change the constraint to $-2xy \le -1$, and the Lagrangian is then

$$\mathcal{L} = 2\ln(x^2+y^2+1) - x^2 - y^2 + \lambda(2xy)$$

The first order conditions (FOC) are

$$\mathcal{L}'_x = \frac{4x}{x^2+y^2+1} - 2x + 2\lambda y = 0$$

$$\mathcal{L}'_y = \frac{4y}{x^2+y^2+1} - 2y + 2\lambda x = 0$$

the constraint (C) is given by $2xy \ge 1$, and the complementary slackness conditions (CSC) are given by

$$\lambda \ge 0 \quad \text{and } \lambda(-2xy+1) = 0$$

To find solutions with $\lambda = 0$ and $2xy = 1$ (binding constraint), we simplify the FOC's using $\lambda = 0$, and get

$$4x - 2x(x^2+y^2+1) = 2x(1-x^2-y^2) = 0$$

and

$$4y - 2y(x^2+y^2+1) = 2y(1-x^2-y^2) = 0$$

Hence $1-x^2-y^2=0$, since $x, y \neq 0$ because of the constraint $2xy = 1$. Finally, $x^2+y^2 = 1$ and $2xy = 1$ gives $x^2 + y^2 = 2xy$, or $x^2 - 2xy + y^2 = 0$. This equation can be written $(x-y)^2 = 0$, which means that $x = y$. The contraint then gives $2x^2 = 1$, or $x^2 = 1/2$ and $x = \pm 1/\sqrt{2}$. Therefore the solutions with $\lambda = 0$ and binding constraint are

$$(x, y; \lambda) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; 0),\ (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}; 0)$$

(b) The admissible set is not bounded. For example, the point $(x, y) = (a, 1/a)$ is admissible for any a since $2xy = 2 \geq 1$. This means that there are admissible points with x arbitrary large, so the set is unbounded. The NDCQ for the constraint $g(x, y) = -2xy \leq -1$ is given by

$$\text{rk}\begin{pmatrix} g'_x & g'_y \end{pmatrix} = \text{rk}\begin{pmatrix} -2y & -2x \end{pmatrix} = 1$$

for the points with $2xy = 1$, and there is no condition when $2xy > 1$. To fail NDCQ, we must have a point with $2xy = 1$ and $-2y = -2x = 0$, and this is clearly not possible. Therefore, no admissible points fail NDCQ.

(c) We must show that the problem has a maximum, and we cannot use the Extreme Value Theorem (the set of admissible points is not bounded) or concavity (it is hard to check if the function $\mathcal{L}(x, y; 0)$ is concave in (x, y) — it gives hard calculations, and it turns out that the function is not concave). We try another approach: We consider the function $f(x, y)$ at a level curve $x^2 + y^2 = c$ for $c \geq 0$, where its value is

$$f(x, y) = 2\ln(x^2 + y^2 + 1) - (x^2 + y^2) = 2\ln(c + 1) - c$$

All points on a given level curve $x^2 + y^2 = c$ therefore have the same value $f(x, y)$, and we consider the function $h(c) = 2\ln(c + 1) - c$ for $c \geq 0$ which measures the value of $f(x, y)$ on this level curve. The derivative is

$$h'(c) = 2/(c + 1) - 1 = (2 - c - 1)/(c + 1) = (1 - c)/(1 + c)$$

Hence the derivative is positive for $c < 1$ and negative for $c > 1$, with $h'(1) = 0$. Therefore $h(c)$ has its maximal value for $c = 1$, and f has its maximal value on the level curve $x^2 + y^2 = 1$. In other words, the unconstrained problem

$$\max f(x, y) = 2\ln(x^2 + y^2 + 1) - x^2 - y^2$$

has global maximum for all points (x, y) with $x^2 + y^2 = 1$ (a circle of radius 1), where $f = 2\ln(2) - 1 \approx 0.386$. If there is an admissible point on this circle, it is therefore a global constrained maximum point, or a solution of the Kuhn-Tucker problem. We consider the conditions

$$2xy \geq 1 \quad \text{and} \quad x^2 + y^2 = 1$$

If $2xy = 1$ and $x^2 + y^2 = 1$, we get exactly the same solutions as in a). These solutions are $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. If $2xy > 1$ and $x^2+y^2 = 1$, then we get $x^2+y^2 = 1 < 2xy$, or $x^2 - 2xy + y^2 < 0$. This is not possible, since it gives $(x - y)^2 < 0$. In conclusion, the two points found in a) are on the level curve $x^2 + y^2 = 1$ and are therefore global maximum points for the Kuhn-Tucker problem, with maximum value $f = 2\ln 2 - 1 \approx 0.386$.