Solutions	Final exam in GRA 6035 Mathematics
Date	December 12th, 2014 at 0900 - 1200

Question 1.

(a) The partial derivatives of $f(x, y, z) = x^4 + y^2 - xz + z^4$ are given by

$$f'_x = 4x^3 - z, \quad f'_y = 2y, \quad f'_z = -x + 4z^3$$

and its Hessian matrix is given by

$$H(f)(x, y, z) = \begin{pmatrix} 12x^2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 12z^2 \end{pmatrix}$$

(b) The stationary points of f are given by

$$f'_x = 4x^3 - z = 0, \quad f'_y = 2y = 0, \quad f'_z = -x + 4z^3 = 0$$

and therefore $y = 0$, $z = 4x^3$, and $-x + 4(4x^3)^3 = -x + 256x^9 = 0$. The last equation gives $x = 0$ or $x^8 = 1/256$, that is $x = \pm 1/2$. From the equation $z = 4x^3$, we see that $x = 0$ gives $z = 0$, $x = 1/2$ gives $z = 1/2$ and $x = -1/2$ gives $z = -1/2$. Therefore there are three stationary points $(x, y, z) = (0, 0, 0)$, $(1/2, 0, 1/2), (-1/2, 0, -1/2)$. The Hessian matrix at these points are the symmetric matrices

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad H(f)\left(\pm\frac{1}{2}, 0, \pm\frac{1}{2}\right) = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

The point $(0, 0, 0)$ is a saddle point since the corresponding Hessian matrix is indefinite; it has a negative second order principal minor $\Delta_2 = -1$ (choose first and third row and column). The points $(1/2, 0, 1/2)$ and $(-1/2, 0, -1/2)$ are local minimum points since the corresponding Hessian matrix is positive definite; it has positive leading principal minors $D_1 = 3$, $D_2 = 6$ and $D_3 = 16$.

(c) The function f is not convex; it if was, all stationary points would be minima, and this is not the case since f has a saddle point. Alternatively, we can consider the leading principal minors $D_1 = 12x^2$, $D_2 = 24x^2$, and $D_3 = 2(144x^2z^2 - 1)$. It is not true that $D_3 \geq 0$ for all (x, y, z), since for instance $D_3(0, 0, 0) = -2$, so f is not convex.

Question 2.

(a) The determinant of A can be developed along the first column:

$$\det(A) = \begin{vmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{vmatrix} = t(t^2 - 1) - 1(t - 1) + 1(1 - t) = t^3 - 3t + 2$$

We can also write $\det(A) = (t - 1)(t(t + 1) - 2) = (t - 1)(t^2 + t - 2) = (t - 1)^2(t + 2)$. When $\lambda = t - 1$, then the matrix $A - \lambda I$ has an echelon form

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore $\text{rk}(A - \lambda I) = 1$.

(b) The matrix A is diagonalizable since it is symmetric. This holds for all t, and therefore also for $t = 8$. Since $\text{rk}(A - \lambda I) = 1$ for $\lambda = t - 1$, if follows that $\det(A - \lambda I) = 0$ and that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has two degrees of freedom. Therefore, $\lambda = t - 1$ is an eigenvalue of multiplicity at least two, and the first two eigenvalues are $\lambda_1 = \lambda_2 = t - 1$. Since $\lambda_1 + \lambda_2 + \lambda_3 = 3t$ (the trace of A), it follows that $\lambda_3 = 3t - 2(t - 1) = t + 2$. When $t = 8$, the eigenvalues are

$\lambda_1 = \lambda_2 = t - 1 = 7$ and $\lambda_3 = t + 2 = 10$.
Alternative. We can also find the eigenvalues by solving the characteristic equation:

$$\begin{vmatrix} 8-\lambda & 1 & 1 \\ 1 & 8-\lambda & 1 \\ 1 & 1 & 8-\lambda \end{vmatrix} = (8-\lambda)((8-\lambda)^2 - 1) - 1(8 - \lambda - 1) + 1(1 - (8 - \lambda))$$

Since $8 - \lambda - 1 = 7 - \lambda$ is a common factor, we can factorize this expression as

$$(7-\lambda)((8-\lambda)(8-\lambda+1) - 1 - 1) = (7-\lambda)(\lambda^2 - 17\lambda + 70)$$

This implies that the eigenvalues are given by $\lambda = 7$, or $\lambda^2 - 17\lambda + 70 = 0$ which gives $\lambda = 7$ or $\lambda = 10$.

(c) If we let s be the share of cars returned to another location, then the share of cars returned to the same location is $8s$. Since $8s + s + s = 1$, we get that $s = 1/10$, and the transition matrix is given by

$$T = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} = 0.1 \begin{pmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{pmatrix} = sA$$

with $s = 1/10$ and $t = 8$. We know from theory that $\lambda = 1$ is an eigenvalue of T (and the dominant eigenvalue in the sense that all other eigenvalues are smaller), and that the long-run equilibrium is the unique eigenvector for T with $\lambda = 1$ such that the components (x, y, z) satisfy $x+y+z = 1$ (that is, the vector can be interpreted as shares of cars). The eigenvectors with $\lambda = 1$ are given by the linear system

$$\begin{pmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

We solve this linear system, for instance using Gaussian elimination, and find that $x = y = z$. The only solution with $x + y + z = 1$ is $x = y = z = 1/3$. It follows that 1/3 of the cars will end up at each location in the long run. Therefore, 40 cars will end up at the airport location.
Alternative. We can also find the eigenvalues of T as $s\lambda$ where $\lambda = 7, 7, 10$ are the eigenvalues of A, since $T = sA$. The eigenvalues of T are therefore $0.7, 0.7, 1$.

Question 3.

(a) The difference equation $y_{t+1} - 3y_t = -5(2t + 1)$ is first order linear, and it has solution $y_t = y_t^h + y_t^p = C \cdot 3^t + y_t^p$ since the $r = 3$is the characteristic root of $r - 3 = 0$. To find a particular solution y_t^p, we consider the right hand side $f_t = -10t - 5$ and the shifted expression $f_{t+1} = -10(t + 1) - 5 = -10t - 15$. We guess a solution $y_t = At + B$. Inserting this guess in the difference equation, we obtain

$$(At + B + A) - 3(At + B) = -10t - 5$$

or $(-2A)t + (A - 2B) = -10t - 5$. We see that $A = 5$ and $B = 5$ is a solution, so $y_t^p = 5t + 5$ and the general solution is

$$y_t = y_t^h + y_t^p = C \cdot 3^t + 5t + 5$$

(b) The differential equation $t^3 y' = y^2$ is separable and it can be written in the form

$$\frac{1}{y^2} y' = \frac{1}{t^3} \quad \Leftrightarrow \quad \int \frac{1}{y^2}\, dy = \int \frac{1}{t^3}\, dt$$

Integration gives $-y^{-1} = -1/2 \cdot t^{-2} + \mathcal{C}$, and therefore that

$$\frac{1}{y} = \frac{1}{2t^2} - \mathcal{C} \quad \text{or} \quad y = \frac{1}{\frac{1}{2t^2} - \mathcal{C}} = \frac{2t^2}{1 - 2\mathcal{C}t^2}$$

(c) The differential equation $(2yt-1)y' = (t+1)e^t - y^2$ can be written in the form $p + qy' = 0$ with

$$p = y^2 - (t+1)e^t, \quad q = 2yt - 1$$

We attempt to find an expression $h = h(y,t)$ such that $h'_t = p$ and $h'_y = q$. From the first equation, we see that $h = ty^2 - te^t + \phi(y)$ since

$$\int (t+1)e^t\,dt = (t+1)e^t - \int 1\cdot e^t\,dt = (t+1)e^t - e^t + \mathcal{C} = te^t + \mathcal{C}$$

using integration by parts with $u = (t+1)$, $v' = e^t$. Using that $h = ty^2 - te^t + \phi(y)$, the second condition $h'_y = q$ becomes

$$h'_y = 2yt + \phi'(y) = 2yt - 1$$

which is satisfied if $\phi'(y) = -1$, or $\phi(y) = -y$. This implies that the equation is exact and that $h = ty^2 - te^t - y$ satisfies $h'_t = p$ and $h'_y = q$. The solution of the differential equation is therefore

$$ty^2 - te^t - y = \mathcal{C} \quad\Leftrightarrow\quad ty^2 - y + (-te^t - \mathcal{C}) = 0$$

To find an explicit solution, we solve for y using the abc-formula:

$$y = \frac{1 \pm \sqrt{1 + 4t(te^t + \mathcal{C})}}{2t}$$

Question 4.

(a) The Kuhn-Tucker problem is already in standard form, so we form the Lagrangian

$$\mathcal{L} = x + 4y + 2z + 5w - \lambda(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2)$$

The first order conditions (FOC) are

$$\begin{aligned}
\mathcal{L}'_x &= 1 - 4\lambda x = 0\\
\mathcal{L}'_y &= 4 - 2\lambda(2y + z) = 0\\
\mathcal{L}'_z &= 2 - 2\lambda(y + 2z) = 0\\
\mathcal{L}'_w &= 5 - 4\lambda w = 0
\end{aligned}$$

the constraint (C) is given by $2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 \le 21$, and the complementary slackness conditions (CSC) are given by

$$\lambda \ge 0 \quad \text{and } \lambda(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 - 21) = 0$$

From the FOC's we see that $\lambda \ne 0$, and therefore $\lambda > 0$ and $2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 = 21$ by the CSC's. The FOC's give

$$x = \frac{1}{4\lambda}, \quad 2y + z = \frac{4}{2\lambda}, \quad y + 2z = \frac{2}{2\lambda}, \quad w = \frac{5}{4\lambda}$$

From the two middle equations, we get that $2y + z = 2(y + 2z)$, and this gives $z = 4z$, or that $z = 0$. We can use this to solve for all variables in terms of λ:

$$x = \frac{1}{4\lambda}, \quad y = \frac{4}{4\lambda}, \quad z = 0, \quad w = \frac{5}{4\lambda}$$

We put these expressions into the constraint, and find that

$$\frac{2\cdot 1^2 + 2\cdot 4^2 + 2\cdot 5^2}{(4\lambda)^2} = 21 \quad \text{or} \quad \frac{84}{16\lambda^2} = \frac{21}{4\lambda^2} = 21$$

which gives $\lambda^2 = 1/4$, or $\lambda = 1/2$ since $\lambda > 0$. Therefore there is only one solution of the Kuhn-Tucker conditions:

$$x = \frac{1}{2},\ y = 2,\ z = 0,\ w = \frac{5}{2},\ \lambda = \frac{1}{2} \quad \text{with} \quad f(x,y,z,w) = 21$$

(b) It follows from the SOC that $(x, y, z, w) = (1/2, 2, 0, 5/2)$ solves the max problem if the function $\mathcal{L}(x, y, z, w; 1/2)$ is concave in (x, y, z, w). We prove that this is the case: The function is given by

$$\mathcal{L}(x, y, z, w; \frac{1}{2}) = x + 4y + 2z + 5w - \frac{1}{2}(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2)$$

Its Hessian matrix is given by

$$H = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The leading principal minors are $D_1 = -2$, $D_2 = 4$, $D_3 = -6$ and $D_4 = 12$. It follows that the Hessian is negative definite, and therefore that $\mathcal{L}(x, y, z, w; 1/2)$ is concave. Hence the candidate point $(x, y, z, w) = (1/2, 2, 0, 5/2)$ is a maximum point by the SOC, with max value $f(1/2, 2, 0, 5/2) = 21$.

(c) We consider the Kuhn-Tucker problem with parameters c, d given by

$$\max\ f(x, y, z, w) = x + cy + 2z + dw \text{ subject to } 2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 \le 21$$

which we have solved for $c = 4$ and $d = 5$. It has Lagrangian

$$\mathcal{L} = x + cy + 2z + dw - \lambda(2x^2 + 2y^2 + 2yz + 2z^2 + 2w^2 - 21)$$

and therefore $\mathcal{L}'_c = y$ and $\mathcal{L}'_d = w$. By the Envelope Theorem, the maximum value changes with approximately

$$\begin{aligned} \Delta f &= \Delta c \cdot \mathcal{L}'_c(x^*, y^*, z^*, w^*; \lambda^*) + \Delta d \cdot \mathcal{L}'_d(x^*, y^*, z^*, w^*; \lambda^*) \\ &= (3.8 - 4) \cdot 2 + (5.4 - 5) \cdot 5/2 = -0.4 + 1.0 = 0.6 \end{aligned}$$

when c changes from $c = 4$ to $c = 3.8$ and d changes from 5 to 5.4, since $y^* = 2$ and $w^* = 5/2$ when $c = 4$ and $w = 5$. The new maximum value is therefore approximately equal to $21 + 0.6 = 21.6$. (The exact value is $7\sqrt{239}/5 = 21.64\ldots$).

Question 5.

Since the sum of the entries in each row, as well as in each column, is 1, it follows that

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{implies that} \quad T \cdot \mathbf{y} = \frac{1}{t + n - 1} \begin{pmatrix} t & 1 & 1 & \ldots & 1 \\ 1 & t & 1 & \ldots & 1 \\ 1 & 1 & t & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

so that $T\mathbf{y} = \mathbf{y}$, or that $\mathbf{y}$ is an eigenvector with eigenvalue $\lambda = 1$. Since the Markov chain is regular, we know from theory that there is an equilibrium state $\mathbf{x}$ that the system will approach as $n \to \infty$,

$$\mathbf{x} = \lim_{n \to \infty} T^n \mathbf{x}_0$$

and this state is the unique eigenvector $\mathbf{x}$ with eigenvalue $\lambda = 1$ such that $x_1 + x_2 + \cdots + x_n = 1$. Since $\mathbf{y}$ is an eigenvector with $\lambda = 1$, it follows that

$$\mathbf{x} = \frac{1}{y_1 + y_2 + \cdots + y_n} \cdot \mathbf{y} = \begin{pmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{pmatrix}$$

is the long run equilibrium state.