BI NORWEGIAN SCHOOL OF MANAGEMENT

Department of Economics

Solutions:	**GRA 60353 Mathematics**
Examination date:	10.12.2010, 09:00 – 12:00
Permitted examination aids:	Bilingual dictionary BI-approved exam calculator: TEXAS INSTRUMENTS BA II Plus^{TM}
Answer sheets:	Squares
Total number of pages:	4

Question 1.

(a) We compute the partial derivatives $f'_x = (x^2 + 2x)e^x$, $f'_y = z$ and $f'_z = y - 3z^2$. The stationary points are given by the equations

$$(x^2 + 2x)e^x = 0, \quad z = 0, \quad y - 3z^2 = 0$$

and this gives $x = 0$ or $x = -2$ from the first equation and $y = 0$ and $z = 0$ from the last two. The stationary points are therefore $(x, y, z) = \boxed{(0,0,0), (-2,0,0)}$.

(b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} (x^2+4x+2)e^x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -6z \end{pmatrix}$$

We see that the second order principal minor obtained from the last two rows and columns is

$$\begin{vmatrix} 0 & 1 \\ 1 & -6z \end{vmatrix} = -1 < 0$$

hence the Hessian is indefinite in all stationary points. Therefore, both stationary points are $\boxed{\text{saddle points}}$.

Question 2.

(a) The determinant of A is given by

$$\det(A) = \begin{vmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{vmatrix} = s(4+2) = \boxed{6s}$$

It follows that the rank of A is 3 if $s \neq 0$ (since $\det(A) \neq 0$). When $s = 0$, A has rank 2 since $\det(A) = 0$ but the minor

$$\begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0$$

Therefore, we get

$$\boxed{\text{rk}(A) = \begin{cases} 3, & s \neq 0 \\ 2, & s = 0 \end{cases}}$$

(b) We compute the characteristic equation of A, and find that

$$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (s-\lambda)(\lambda^2 - 5\lambda + 6) = (s-\lambda)(\lambda-2)(\lambda-3)$$

Therefore, the eigenvalues of A are $\boxed{\lambda = s, 2, 3}$. Furthermore, we have that

$$A\mathbf{v} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix}$$

We see that $\mathbf{v}$ is an eigenvector for A if and only if $\boxed{s = 6}$, in which case $A\mathbf{v} = 6\mathbf{v}$.

(c) If $s \neq 2, 3$, then A has three distinct eigenvalues, and therefore A is diagonalizable. If $s = 2$, we check the eigenspace corresponding to the double root $\lambda = 2$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. If $s = 3$, we check the eigenspace corresponding to the double root $\lambda = 3$: The coefficient matrix of the system has echelon form

$$\begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank two, so there is only one free variable. In both cases, there are too few linearly independent eigenvectors, and A is not diagonalizable. Hence A is diagonalixable if $\boxed{s \neq 2, 3}$.

Question 3.

(a) We have $b_{t+1} - b_t = rb_t - s_{t+1}$, where $s_{t+1} = 500 + 10t$ is the repayment in period $t+1$. Hence we get the difference equation

$$b_{t+1} = (1+r)b_t - (500 + 10t), \quad b_0 = K$$

The homogenous solution is $b_t^h = C(1+r)^t$. We try to find a particular solution of the form $b_t = At + B$, which gives $b_{t+1} = At + A + B$. Substitution in the difference equation gives

$$At + A + B = (1+r)(At+B) - (500 + 10t) = ((1+r)A - 10)t + (1+r)B - 500$$

and this gives $A = 10/r$ and $B = 10/r^2 + 500/r$. Hence the solution of the difference equation is

$$b_t = b_t^h + b_t^p = C(1+r)^t + \frac{10}{r}t + \frac{10}{r^2} + \frac{500}{r}$$

The initial value condition is $K = C + 10/r^2 + 500/r$, hence we obtain the solution

$$b_t = \boxed{\left(K - \frac{10}{r^2} - \frac{500}{r}\right)(1+r)^t + \frac{10}{r}t + \frac{10}{r^2} + \frac{500}{r}}$$

(b) The homogeneous equation $y'' + y' - 6y = 0$ has characteristic equation $r^2 + r - 6 = 0$ and roots $r = 2$ and $r = -3$, so $y_h = C_1 e^{2t} + C_2 e^{-3t}$. We try to find a particular solution of the form $y = (At + B)e^t$, which gives

$$y' = (At + A + B)e^t, \quad y'' = (At + 2A + B)e^t$$

Substitution in the differential equation gives

$$(At + 2A + B)e^t + (At + A + B)e^t - 6(At + B)e^t = te^t \Leftrightarrow -4A = 1 \text{ and } 3A - 4B = 0$$

This gives $A = -1/4$ and $B = -3/16$. Hence the general solution of the differential equation is $y = y_h + y_p = \boxed{C_1 e^{2t} + C_2 e^{-3t} - (\frac{1}{4}t + \frac{3}{16})e^t}$

(c) The differential equation can be written in the form

$$\left(3t^2 - \frac{1}{y}\right) + \frac{t}{y^2}y' = 0$$

and we see that it is exact. Hence it can be written of the form $u(y,t) = C$, where $u(y,t)$ is a function that satisfies

$$\frac{\partial u}{\partial t} = 3t^2 - \frac{1}{y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{t}{y^2}$$

One solution is $u(y,t) = t^3 - t/y$, and this gives

$$t^3 - \frac{t}{y} = C \quad \Leftrightarrow \quad y = \frac{t}{t^3 - C}$$

The initial condition gives $1/(1-C) = 1/3$ or $C = -2$. The solution to the initial value problem is therefore

$$y = \boxed{\frac{t}{t^3+2}}$$

QUESTION 4.

(a) We compute the Hessian of g, and find

$$g'' = \frac{1}{xyz}\begin{pmatrix} \frac{2}{x^2} & \frac{1}{xy} & \frac{1}{xz} \\ \frac{1}{xy} & \frac{2}{y^2} & \frac{1}{yz} \\ \frac{1}{xz} & \frac{1}{yz} & \frac{2}{z^2} \end{pmatrix}$$

Hence the leading principal minors are

$$D_1 = \frac{1}{xyz}\frac{2}{x^2} > 0, \quad D_2 = \frac{1}{(xyz)^2}\frac{3}{(xy)^2} > 0, \quad D_3 = \frac{1}{(xyz)^3}\frac{4}{(xyz)^2} > 0$$

This means that g is $\boxed{\text{convex}}$.

(b) The set $\{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ is closed and bounded, so the problem has solutions by the extreme value theorem. The NDCQ is satisfied, since the rank of $\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = 1$ when $x^2+y^2+z^2 = 1$. We form the Lagrangian

$$\mathcal{L} = xyz - \lambda(x^2+y^2+z^2-1)$$

and solve the Kuhn-Tucker conditions, consisting of the first order conditions

$$\mathcal{L}'_x = yz - \lambda \cdot 2x = 0$$
$$\mathcal{L}'_y = xz - \lambda \cdot 2y = 0$$
$$\mathcal{L}'_z = xy - \lambda \cdot 2z = 0$$

together with one of the following conditions: i) $x^2+y^2+z^2 = 1$ and $\lambda \geq 0$ or ii) $x^2+y^2+z^2 < 1$ and $\lambda = 0$. We first solve the equations/inequalities in case i): If $x = 0$, then we see that $y = 0$ or $z = 0$ from the first equation, and we get the solutions $(x,y,z;\lambda) = (0,0,\pm1;0), (0,\pm1,0;0)$. If $x \neq 0$, we get $2\lambda = yz/x$ and the remaining first order conditions give $(x^2-y^2)z = 0$ and $(x^2-z^2)y = 0$. If $y = 0$, we get the solution $(\pm1,0,0;0)$. Otherwise, we get $x^2 = y^2 = z^2$, hence

$$(x,y,z;\lambda) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{2\sqrt{3}}\right)$$

The condition that $\lambda \geq 0$ give that either all three coordinates (x,y,z) are positive, or that one is positive and two are negative. In total, we obtain four different solutions. We note that $f(x,y,z) = \frac{1}{3\sqrt{3}}$ for each of these four solutions, while $f(x,y,z) = 0$ for either of the first three solutions. Finally, we consider case ii), where $\lambda = 0$. This gives $xy = xz = yz = 0$, and we obtain

$$(x,y,z;\lambda) = (a,0,0;0), (0,a,0;0), (0,0,a;0)$$

The condition that $x^2 + y^2 + z^2 < 1$ give $a^2 \leq 1$ or $a \in (-1, 1)$. For all these solutions, we get $f(x, y, z) = 0$. We can therefore conclude that the solution to the optimization problem is a maximum value of

$$\boxed{\frac{1}{3\sqrt{3}}}$$