Proof of SOC:

a) Lagrange case:

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

SOC

If $(x_1^*, x_2^*, .., x_n^*; \lambda_1^*, .., \lambda_m^*)$ satisfy FOC+C, and the function

$$h(x_1, .., x_n) = \mathcal{L}(x_1, .., x_n; \lambda_1^*, .., \lambda_m^*)$$

is concave, then $\underline{x}^*$ is max, and if $h(\underline{x})$ is convex, then $\underline{x}^*$ is min.

Proof:

Assume that $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC+C and $h(\underline{x})$ is concave. Then $\underline{x}^*$ is stationary pt for h, since it is a solution to the equations $\partial h/\partial x_i = 0$. This follows since FOC can be written

$$\left.\begin{aligned} \frac{\partial h}{\partial x_1}(\underline{x}^*) &= \frac{\partial \mathcal{L}}{\partial x_1}(\underline{x}^*; \underline{\lambda}^*) = 0 \\ \frac{\partial h}{\partial x_2}(\underline{x}^*) &= \frac{\partial \mathcal{L}}{\partial x_2}(\underline{x}^*; \underline{\lambda}^*) = 0 \\ &\vdots \\ \frac{\partial h}{\partial x_n}(\underline{x}^*) &= \frac{\partial \mathcal{L}}{\partial x_n}(\underline{x}^*; \underline{\lambda}^*) = 0 \end{aligned}\right\} \text{ holds since } (\underline{x}^*; \underline{\lambda}^*) \text{ satisfy FOC's}$$

A stationary pt of h is global max since h is concave. This means in particular that for any pt $\underline{x}$ that satisfy C, we have

$$h(\underline{x}^*) \geq h(\underline{x})$$

$$\mathcal{L}(\underline{x}^*; \underline{\lambda}^*) \geq \mathcal{L}(\underline{x}; \underline{\lambda}^*)$$

$$f(\underline{x}^*) - \sum_i \lambda_i^* g_i(\underline{x}^*) \geq f(\underline{x}) - \sum_i \lambda_i^* g_i(\underline{x})$$

The last inequality means that $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ that satisfies C, since

$$\sum \lambda_i^* g_i(x^*) = \sum \lambda_i^* a_i = \sum \lambda_i^* g_i(\underline{x})$$

$\left\{ \begin{array}{l} g_i(\underline{x}^*) = a_i \\ \text{and} \\ g_i(\underline{x}) = a_i \end{array} \right.$ since both $\underline{x}^*, \underline{x}$ satisfy C

This implies that $\underline{x}^*$ is maximal among admissible pts, ie a solution to the max problem.

For the min problem, the proof is similar. If h is convex and $\underline{x}^*$ stationary for h, then $\underline{x}^*$ is global min for h. The result follows by turning all inequalities from $\geq$ to $\leq$. □

b) Kuhn - Tucker case: max $f(\underline{x})$ when $\left\{ \begin{array}{l} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{array} \right.$

SOC:

If $(\underline{x}^*, \underline{\lambda}^*)$ satisfy FOC+C+CSC, and the function

$$h(\underline{x}) = \mathcal{L}(\underline{x}; \underline{\lambda}^*)$$

is concave, then $\underline{x}^*$ is max.

Proof:

Assume $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC+C+CSC, and that h is concave. The $\underline{x}^*$ is stationary for h since $(\underline{x}^*; \underline{\lambda}^*)$ satisfy FOC (same proof as in Lagrange case). So $\underline{x}^*$ is global max for h.

For any $\underline{x}$ that satisfy C, we have

$$h(\underline{x}^*) \geq h(\underline{x})$$

$$f(\underline{x}^*) - \sum_i \lambda_i^* g_i(\underline{x}^*) \geq f(\underline{x}) - \sum_i \lambda_i^* g_i(\underline{x})$$

$$f(\underline{x}^*) \geq f(\underline{x})$$

For each i, either $\lambda_i^* = 0$, or $\lambda_i^* > 0$ and $g_i(\underline{x}^*) = g_i(\underline{x}) = a_i$ since $\underline{x}^*, \underline{x}$ satisfy C + CSC. In either case $\lambda_i^* g_i(\underline{x}^*) = \lambda_i^* g_i(\underline{x})$

This implies $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ that satisfy C, so $\underline{x}^*$ solves max problem. □

Remark:

In both Lagrange / Kuhn-Tucker case, the essential point is not that h is concave but that $\underline{x}^*$ is a global max for h.

In other words, if $h(\underline{x}) = h(\underline{x}; \underline{\lambda}^*)$ is not concave, but $\underline{x}^*$ is still a max for $h(\underline{x})$, then the conclusion of SOC still holds.