Alternative method: NDCQ

The alternative method is described in [ME] 19.5, and works for both Lagrange and Kuhn-Tucker problems.

Ex: max y when $x^2+y^3=0$

Alternative method:

Use the Lagrangian

$$\mathcal{L} = \lambda_0 \cdot y - \lambda_1 \cdot (x^2+y^3)$$

FOC:

$$\mathcal{L}'_x = -\lambda_1 \cdot 2x = 0$$
$$\mathcal{L}'_y = \lambda_0 - \lambda_1 \cdot 3y^2 = 0$$

C:

$$x^2+y^3=0$$

Note the new Lagrangian, with an extra multiplier λ_0. In addition to the usual conditions, we add

i) $\lambda_0 = 0$ or $\lambda_0 = 1$

ii) $(\lambda_0, \lambda_1) \neq (0,0)$

Original method:

$$\mathcal{L} = y - \lambda_1 (x^2+y^3)$$

$\mathcal{L}'_x = -\lambda_1 \cdot 2x = 0$

$\mathcal{L}'_y = 1 - \lambda_1 \cdot 3y^2 = 0$

$x^2+y^3=0$

$\lambda_1 = 0$ or $x = 0$
$\lambda_1 = 0$ not poss.
$\Rightarrow \lambda_1 \neq 0, x = 0$
$y = 0$
$\Downarrow$
not poss. since
$1 - \lambda_1 3y^2 = 1 \neq 0$
$\Downarrow$
no solution of FOC + C

NDCQ:

$\operatorname{rk} \begin{pmatrix} 2x & 3y^2 \end{pmatrix} = 1$

NDCQ fails:

$2x = 3y^2 = 0 \Rightarrow x = y = 0$ which is admissible

Solution $(x,y) = (0,0)$ of NDCQ fails + C

Solution:

$-\lambda_1 \cdot 2x = 0$ gives $x=0$ or $\lambda_1 = 0$

$\lambda_1 = 0$ means $\lambda_0 = 0$ from second FOC

$\Rightarrow (\lambda_0, \lambda_1) = (0,0)$ not possible

$\Rightarrow \lambda_1 \neq 0, x = 0 \Rightarrow y = 0$ by constr. $\Rightarrow \lambda_0 = 0$

$\Bigg\}$ $(x,y;\lambda_0,\lambda_1) = (0,0;0,\lambda_1)$ with $\lambda_1 \neq 0$

$f = y = 0.$

In alternative method:	Original method
Solutions with $\lambda_0 = 1$ $\longrightarrow$	Solutions of FOC + C
— " — $\lambda_0 = 0$ $\longrightarrow$	Solutions where NDCQ fails + C (ie. adm. pts where NDCQ fails)

General result for Lagrange problems:

Problem: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Form the Lagrangian $\mathcal{L}(\underline{x}; \lambda_0, \lambda_1, .., \lambda_m) = f(\underline{x}) \cdot \lambda_0 - \lambda_1 \cdot g_1(x) - \dots - \lambda_m g_m(x)$

Consider the conditions

i) $\mathcal{L}'_{x_1} = 0$ $\mathcal{L}'_{x_2} = 0$ $\cdots$ $\mathcal{L}'_{x_n} = 0$

ii) $g_1(\underline{x}) = a_1$ $g_2(\underline{x}) = a_2$ $\cdots$ $g_m(\underline{x}) = a_m$

iii) $\lambda_0 = 0$ or $\lambda_0 = 1$, and $(\lambda_0, \lambda_1, .., \lambda_m) \neq (0, 0, .., 0)$

If the problem has a solution $\underline{x}^*$, the there are multipliers λ_0^*, $\lambda_1^*, .., \lambda_m^*$ such that conditions i) → iii) hold.

General result for Kuhn-Tucker problems:

Problem: max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

Form the Lagrangian $\mathcal{L}(\underline{x}; \lambda_0, .., \lambda_m) = f(x) \cdot \lambda_0 - \lambda_1 \cdot g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$.

Consider the conditions

i) $\mathcal{L}'_{x_1} = 0$ $\mathcal{L}'_{x_2} = 0$ $\cdots$ $\mathcal{L}'_{x_n} = 0$

ii) $g_1(\underline{x}) \le a_1$ $g_2(\underline{x}) \le a_2$ $\cdots$ $g_m(\underline{x}) \le a_m$

iii) $\lambda_1 \ge 0, \lambda_2 \ge 0, .., \lambda_m \ge 0$ and $\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0, \dots, \lambda_m(g_m(\underline{x}) - a_m) = 0$

iv) $\lambda_0 = 0$ or $\lambda_0 = 1$, $(\lambda_0, \lambda_1, .., \lambda_m) \neq (0, 0, .., 0)$

If the problem has a solution $\underline{x}^*$, then there are multipliers $\lambda_0^*, \lambda_1^*, \dots, \lambda_m^*$ such that conditions i) - iv) hold.

<u>Conclusion:</u>

In both Lagrange / Kuhn-Tucker problems, the candidates for max / min are <u>only</u> the solutions to the conditions i) - iii) or i) - iv) above.

It is <u>not</u> necessary to consider NDCQ seperately when you use $\mathcal{L}$ with extra multiplier λ_0; admissible pts where NDCQ fails come out as candidates with $\underline{\lambda_0 = 0}$.

<u>Ex:</u> max xy when $x^2+y^2 \leq 1$

$\mathcal{L} = \lambda_0 \cdot xy - \lambda_1(x^2+y^2)$

$$\left.\begin{aligned} \mathcal{L}'_x &= \lambda_0 y - \lambda_1 \cdot 2x = 0 \\ \mathcal{L}'_y &= \lambda_0 x - \lambda_1 \cdot 2y = 0 \end{aligned}\right\} \text{New FOC's}$$

$$x^2+y^2 \leq 1 \quad \} \; C$$

$$\lambda_1 \geq 0, \quad \lambda_1(x^2+y^2-1) = 0 \quad \} \; CSC$$

$$\lambda_0 = 0 \text{ or } \lambda_0 = 1, \quad (\lambda_0, \lambda_1) \neq (0,0) \quad \} \text{ New cond. on } \lambda_0$$

a) $\underline{\lambda_0 = 0}$: $\lambda_1 \neq 0$ since $(\lambda_0, \lambda_1) \neq (0,0)$

FOC: $x=0$, $y=0$ C: ok

CSC: $\lambda_1 \cdot (-1) = 0 \Rightarrow \lambda_1 = 0$ <u>not possible</u>

b) $\underline{\lambda_0 = 1}$: (Used FOC+C+CSC)

FOC: $y = 2\lambda_1 x$

$x = 2\lambda_1 \cdot y = 2\lambda_1(2\lambda_1 x)$

$x(1-4\lambda_1^2) = 0$

$\underline{x=0}$ or $1 = 4\lambda_1^2$

$\lambda_1^2 = 1/4$

$\underline{\lambda_1 = 1/2}$ since $\lambda_1 \geq 0$

If $\underline{x=0}$: $y=0 \Rightarrow \lambda_1 = 0 \Rightarrow$ Cand: $(0,0;0)$ $f=0$

If $\underline{x \neq 0, \lambda_1 = 1/2}$: $y = x$, $x^2+y^2 = 1 \Rightarrow x^2 = y^2 = 1/2$

$\Rightarrow$ Cand: $(\pm\sqrt{1/2}, \pm\sqrt{1/2}; 1/2)$ $f = \pm 1/2$

<u>Conclusion:</u>

<u>Best cand:</u>

$$\left.\begin{aligned} &(\sqrt{1/2}, \sqrt{1/2}; 1/2) \quad f = 1/2 \\ &(-\sqrt{1/2}, -\sqrt{1/2}, 1/2) \quad f = 1/2 \end{aligned}\right\} \lambda_0 = 1$$

since <u>no more candidates</u> (not necessary to check NDCQ)

$x^2+y^2 \leq 1$ is bounded, so <u>this</u> is <u>max</u>