GRA 6035 Mathematics

Problems for Lecture 12

Key problems

Problem 1.
Write the systems of differential equations on matrix form and solve them:
a) $y_1' = 2y_1 - 5y_2$ and $y_2' = -5y_1 + 2y_2$ *b)* $y_1' = y_2$ and $y_2' = 4y_1 + 3y_2$

Problem 2.
Solve the systems of differential equations:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Problem 3.
Find all equilibrium states in Problem 2. Are there globally asymptotically stable equilibrium states?

Problems from Differential Equations

Exercise problems 2.1 - 2.6 (full solutions on the web page)

Exam problems

Mock exam 11/2017 1 - 3 (full solutions on the web page)

Problems from the Digital Workbook

Exercise problems 12.1 - 12.13 (full solutions in the workbook)

Answers to key problems

Problem 1.

$$a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{-3t} - C_2 e^{7t} \\ C_1 e^{-3t} + C_2 e^{7t} \end{pmatrix} \quad b) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{4t} - C_2 e^{-t} \\ 4C_1 e^{4t} + C_2 e^{-t} \end{pmatrix}$$

Problem 2.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} C_1 e^{-4t} & & - C_2 e^{-6t} \\ & C_2 e^{-3t} & \\ C_1 e^{-4t} & & + C_2 e^{-6t} \end{pmatrix}$$

Problem 3.
There is one equilibrium state $\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, and it is globally asymptotically stable since all eigenvalues are negative.

Solutions: Key problems Lecture 12

1. a) $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $A = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}$

Eigenvalues and eigenvectors of A:

$\begin{vmatrix} 2-\lambda & -5 \\ -5 & 2-\lambda \end{vmatrix} = 0$ $\lambda^2 - 4\lambda - 21 = 0$

$\lambda = \dfrac{4 \pm \sqrt{16-4(-21)}}{2} = \dfrac{4 \pm 10}{2}$

$\lambda_1 = \underline{-3}, \ \lambda_2 = \underline{7}$

E_{-3}: $\lambda = -3$

$\begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix} \to \begin{pmatrix} 5 & -5 \\ 0 & 0 \end{pmatrix}$ $5x - 5y = 0 \Rightarrow x = y$
y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $E_{-3} = \text{span}(\underline{v_1})$

E_7: $\lambda = 7$

$\begin{pmatrix} -5 & -5 \\ -5 & -5 \end{pmatrix} \to \begin{pmatrix} -5 & -5 \\ 0 & 0 \end{pmatrix}$ $-5x - 5y = 0 \Rightarrow x = -y$
y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $E_7 = \text{span}(\underline{v_2})$

General solution:

$$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{7t}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-3t} - c_2 e^{7t} \\ c_1 \cdot e^{-3t} + c_2 e^{7t} \end{pmatrix}$$

b) $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\qquad A = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$

$\begin{vmatrix} -\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 3\lambda - 4 = 0$

$\lambda_1 = -1, \ \lambda_2 = 4$

$\underline{E_{-1}}$: $\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \to \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad E_{-1} = \text{span}(\underline{v_1})$

$\underline{E_4}$: $\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix} \to \underline{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad E_4 = \text{span}(\underline{v_2})$

General solution:

$\underline{y} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -c_1 e^{-t} + c_2 e^{4t} \\ c_1 e^{-t} + 4 c_2 e^{4t} \end{pmatrix}$

c) $A = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix}$: $\qquad \begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = 0$

$(-3-\lambda) \cdot (\lambda^2 + 10\lambda + 24) = 0$

$\lambda = -3$ or $\lambda^2 + 10\lambda + 24 = 0$

$\lambda_1 = -3, \quad \lambda_2 = -4, \quad \lambda_3 = -6$

$\underline{E_{-3}}$:

$\begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $x - 2z = 0 \Rightarrow x = 0$, $-3z = 0 \Rightarrow z = 0$, y free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{E_{-4}}$:

$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $-x + z = 0$, $y = 0$, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{E_{-6}}$:

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ $x + z = 0$, $3y = 0$, z free $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

General solution:

$$\underline{y} = C_1 \cdot \underline{v}_1 e^{\lambda_1 t} + C_2 \cdot \underline{v}_2 e^{\lambda_2 t} + C_3 \underline{v}_3 e^{\lambda_3 t}$$

$$= C_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-3t} + C_2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-4t} + C_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-6t}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} C_2 e^{-4t} - C_3 e^{-6t} \\ C_1 e^{-3t} \\ C_2 e^{-4t} + C_3 e^{-6t} \end{pmatrix}$$

3. Eq. states:

$$\underline{y}' = 0 \Rightarrow A\underline{y} = \underline{0}$$

$$\underline{y} = A^{-1} \cdot \underline{0}$$

$$\underline{y}_e = \underline{0}$$

$$|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = (-3)(-4)(-6) = -72 \neq 0$$

$\Downarrow$

A^{-1} exists

$$\underline{y}_e = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Stability: $\lambda_1, \lambda_2, \lambda_3 < 0 \Rightarrow \underline{y}_e = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is globally asymptotically stable

$(e^{-3t}, e^{-4t}, e^{-6t} \to 0$ as $t \to \infty)$