GRA 6035 Mathematics

Problems for Lecture 11

Key problems

Problem 1.
Solve the exact differential equations:
a) $3t^2 - 2t + 2y \cdot y' = 0$ *b)* $2y - 3t^2 + 2(y+t)y' = 0$ *c)* $\dfrac{y(1-2\ln t)}{t^3} + \dfrac{\ln t}{t^2} \cdot y' = 0$

Problem 2.
Find the equilibrium states and determine their stability:
a) $y' = 6 - 2y$ *b)* $y' = y^2 - 4$ *c)* $y' = 5y(1 - y/10)$
In each case, sketch solutions curves in the (t, y)-coordinate system to illustrate the results.

Problem 3.
Solve the differential equations:
a) $y'' + 6y' - 16y = 16t - 22$ *b)* $y'' + 6y' + 9y = 4e^{-t}$ *c)* $y'' - 3y' + 2y = 3e^{2t}$ *d)* $y'' - y = t^2$

Problems from Differential Equations

Exercise problems	1.17 - 1.34 (full solutions on the web page)

Problems from the Digital Workbook

Exercise problems	10.13 - 10.16, 11.1 - 11.16 (full solutions in the workbook)
Excel problems	11.17 (full solutions in the workbook) As a minimum, you should understand what happens when you change the parameters in the Excel models that are available in the workbook.

Answers to key problems

Problem 1.
a) $y = \pm\sqrt{t^2 - t^3 + C}$ *b)* $y = -t \pm \sqrt{t^2 + t^3 + C}$ *c)* $y = \dfrac{Ct^2}{\ln t}$

Problem 2.
a) $y_e = 3$ is globally asymptotically stable
b) $y_e = -2$ is stable (but not globally asymptotically stable), $y_e = 2$ is unstable
c) $y_e = 0$ is unstable, $y_e = 10$ is stable (but not globally asymptotically stable)

Problem 3.
a) $y = C_1 e^{-8t} + C_2 e^{2t} + 1 - t$ *b)* $y = C_1 e^{-3t} + C_2 te^{-3t} + e^{-t}$ *c)* $y = C_1 e^{2t} + C_2 e^t + 3te^{2t}$ *d)* $y = C_1 e^t + C_2 e^{-t} - t^2 - 2$

Solutions: Key problems Lecture 11

1. a) $3t^2-2t+2y\cdot y'=0$

Exact? $p=3t^2-2t \;= h'_t$

$q=2y \;= h'_y$

$\Downarrow$

I: $h=\int 3t^2-2t\,dt = t^3-t^2+g(y)$

II: $h'_y=0+g'(y)=2y \Rightarrow g(y)=y^2+c=y^2 \leftarrow$ may choose $c=0$

$\Downarrow$

$h=t^3-t^2+y^2$ satisfies I·II, eqn. is exact

Solution: $t^3-t^2+y^2=C \leftarrow h(t,y)=C$

$y^2=C+t^2-t^3$

$y=\pm\sqrt{C+t^2-t^3}$

b) $2y-3t^2+2(y+t)y'=0$

$p=2y-3t^2 = h'_t \Rightarrow h=2yt-t^3+g(y)$

$q=2y+2t = h'_y \qquad h'_y=2t-0+g'(y)=2y+2t$

$g'(y)=2y$

$g(y)=y^2+c=y^2$

$h=2yt-t^3+y^2$ exact

$2yt-t^3+y^2=C$

$y^2+2t\cdot y+(-t^3-c)=0$

$$y=\frac{-2t\pm\sqrt{4t^2-4\cdot(-t^3-c)}}{2}$$

$$=-t\pm\sqrt{t^2+t^3+C}$$

c) $\dfrac{y(1-2\ln t)}{t^3} + \dfrac{\ln t}{t^2} y' = 0$

$p = \dfrac{y(1-2\ln t)}{t^3} \quad = h'_t$

Start with second eqn:

$q = \dfrac{\ln t}{t^2} \quad = h'_y \quad \Rightarrow h = \dfrac{\ln t}{t^2} \cdot y + g(t)$

$h'_t = \dfrac{\frac{1}{t} \cdot t^2 - \ln t \cdot 2t}{t^4} y + g'(t)$

$= \dfrac{t - 2t\ln t}{t^4} y + g'(t)$

$p = y \dfrac{(1-2\ln t)}{t^3} \quad \xleftarrow[\text{equal } p]{\text{should}} \quad = \dfrac{1-2\ln t}{t^3} y + g'(t)$

ok if $g'(t) = 0$

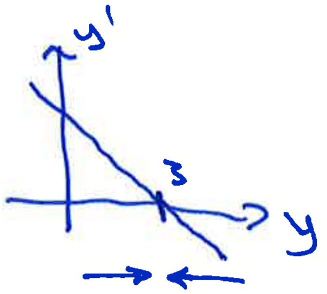
choose $\underline{g(t) = 0}$

$h = \dfrac{\ln t}{t^2} \cdot y = c$

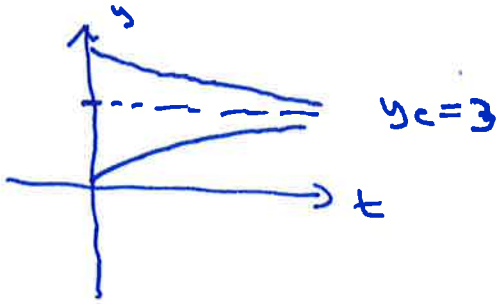
$$y = \underline{\underline{\dfrac{c t^2}{\ln t}}}$$

2. a) $y' = 6 - 2y$

Eq. state: $6 - 2y = 0$
$y = 3$ $\Rightarrow$ $y_e = 3$



phase diagram

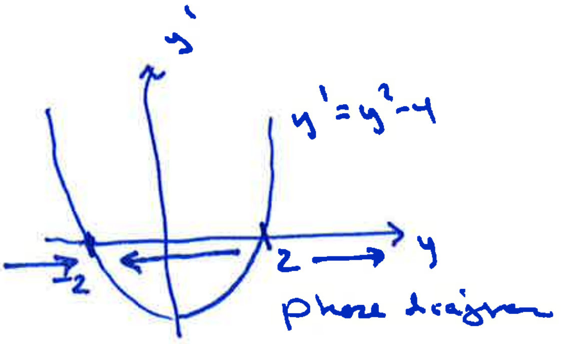


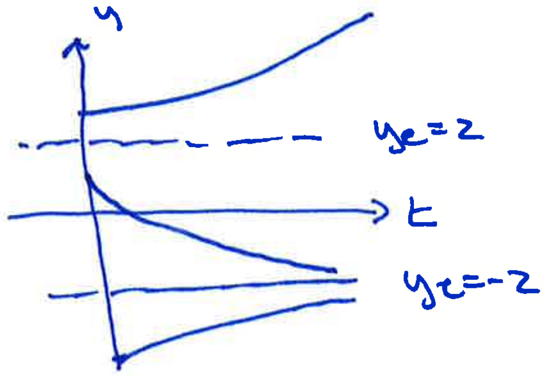
$y_e = 3$ is globally asymptotically stable

b) $y' = y^2 - 4$

Eq. state: $y^2 - 4 = 0$
$y = \pm 2$

Eq. states:
$y_e = -2$ and $y_e = 2$





$y_e = 2$ is unstable

$y_e = -2$ is stable, but not globally asymptotically stable

Alternative method: Stability thm.

$F(y) = y^2 - 4$
$F'(y) = 2y$

$F'(2) = 4 > 0 \Rightarrow y_e = 2$ unstable
$F'(-2) = -4 < 0$ $\quad y_e = -2$ stable

note: cannot tell if $y_e = -2$ is globally asymptotically stable from $F'(-2)$.

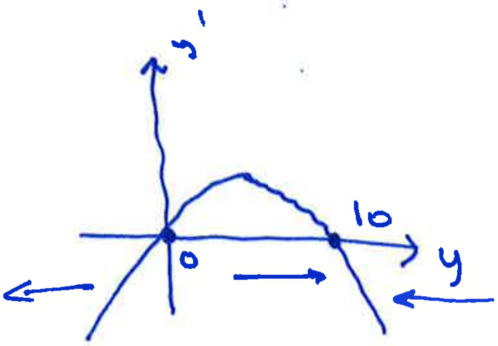
c) $y' = 5y(1 - y/10)$

<u>Eq. state:</u>

$5y(1 - y/10) = 0$

$y = 0$ or $1 - y/10 = 0$

$y = 10$



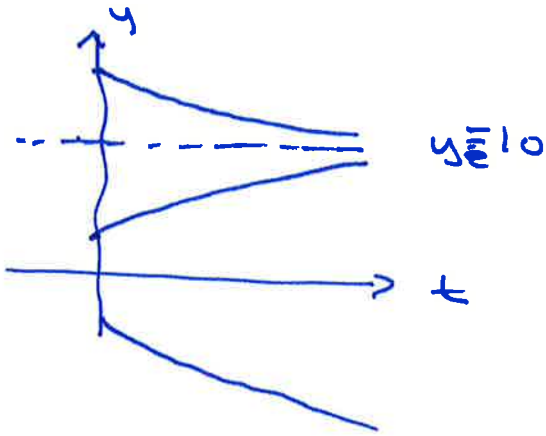
phase diagram

$y_e = 0$ <u>unstable</u>

$y_e = 10$ <u>stable</u>, but not globally asymptotically stable

Eq. states:

<u>$y_e = 0$</u> and <u>$y_e = 10$</u>



3. a) $y'' + 6y' - 16y = 16t - 22$

$$y = y_h + y_p = \underline{\underline{C_1 e^{2t} + C_2 e^{-8t} - t + 1}}$$

$\underline{y_h}$: $y'' + 6y' - 16y = 0$

$r^2 + 6r - 16 = 0$

$r = 2,\ r = -8 \Rightarrow y_h = \underline{C_1 e^{2t} + C_2 e^{-8t}}$

$\underline{y_p}$: $y'' + 6y' - 16y = 16t - 22 \leftarrow$ $f(t) = 16t - 22$

$f'(t) = 16$

$f''(t) = 0$

$0 + 6 \cdot A - 16(At + B) = 16t - 22 \leftarrow$

Guess:

$y = \underline{At + B}$

$y' = A$

$y'' = 0$

$$(\underbrace{-16A}_{\text{coeff. of } t})t + (\underbrace{6A - 16B}_{\text{const.}}) = 16t - 22$$

<u>Comparison of coeff's:</u>

$-16A = 16 \Rightarrow A = \underline{-1}$

$6A - 16B = -22$

$-6 - 16B = -22$

$-16B = -16$

$\underline{B = 1}$

$$y_p = At + B = \underline{-t + 1}$$

b) $y'' + 6y' + 9y = 4e^{-t}$

$$y = y_n + y_p = \underline{\underline{C_1 e^{-3t} + C_2 t e^{-3t} + e^{-t}}}$$

$\underline{y_n}$: $y'' + 6y' + 9y = 0$

$$r^2 + 6r + 9 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$$

$$= -3 \pm 0$$

$$r_1 = r_2 = \underline{-3} \quad \text{(double root)} \Longrightarrow y_n = \underline{C_1 e^{-3t} + C_2 t e^{-3t}}$$

y_p: $y'' + 6y' + 9y = 4e^{-t}$

$$\left.\begin{aligned} f &= 4e^{-t} \\ f' &= -4e^{-t} \\ f'' &= 4e^{-t} \end{aligned}\right\} \quad \begin{aligned} y &= Ae^{-t} \leftarrow \underline{\text{guess}} \\ y' &= -Ae^{-t} \\ y'' &= Ae^{-t} \end{aligned}$$

$$(Ae^{-t}) + 6(-Ae^{-t}) + 9(Ae^{-t}) = 4e^{-t}$$

$$(A - 6A + 9A)e^{-t} = 4e^{-t}$$

$$(4A)e^{-t} = 4e^{-t}$$

$\underline{\text{Comparison of coeff's:}}$

$$4A = 4$$

$$A = 1$$

$$y_p = Ae^{-t} = \underline{e^{-t}}$$

c) $y'' - 3y' + 2y = 3e^{2t}$

$y = y_h + y_p = \underline{\underline{C_1 e^{t} + C_2 e^{2t} + 3te^{2t}}}$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 1, r = 2 \implies y_h = \underline{C_1 e^{t} + C_2 e^{2t}}$

y_p: $y'' - 3y' + 2y = 3e^{2t}$

$f = 3e^{2t}$ $\qquad y = Ae^{2t}$ ← guess

$f' = 6e^{2t}$ $\qquad y' = 2Ae^{2t}$

$f'' = 12e^{2t}$ $\qquad y'' = 4Ae^{2t}$

$(4Ae^{2t}) - 3(2Ae^{2t}) + 2(Ae^{2t}) = 3e^{2t}$

$(4A - 6A + 2A)e^{2t} = 3e^{2t}$

$0A e^{2t} = 3e^{2t}$ ← no solution

try $y = t \cdot Ae^{2t} = \underline{Ate^{2t}}$

$y' = Ae^{2t} + At \cdot e^{2t} \cdot 2$

$= (\underline{A + 2At)e^{2t}}$

$y'' = 2Ae^{2t} + (A + 2At)e^{2t} \cdot 2$

$= \underline{(4A + 4At)e^{2t}}$

$(4A + 4At)e^{2t} - 3(A + 2At)e^{2t} + 2 \cdot (At)e^{2t} = 3e^{2t}$

$(4A - 3A + \cancel{4At - 6At + 2At})e^{2t} = 3e^{2t}$

$Ae^{2t} = 3e^{2t}$

$\underline{A = 3}$

$y_p = Ate^{2t} = \underline{3te^{2t}}$

d) $y'' - y = t^2$

$y = y_h + y_p = \underline{\underline{c_1 e^t + c_2 e^{-t} - t^2 - 2}}$

$\underline{y_h:}$ $y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1 \implies y_h = \underline{c_1 e^t + c_2 e^{-t}}$

$\underline{y_p:}$ $y'' - y = t^2$ $\quad f = t^2$ $\quad$ $\underline{\text{Guess:}}$ $y = At^2 + Bt + C$

$f' = 2t$ $\quad$ $y' = 2At + B$

$f = 2$ $\quad$ $y'' = 2A$

$2A - (At^2 + Bt + C) = t^2$

$(-A)t^2 + (-B)t + (2A - C) = t^2$

$\underline{\text{Comparison of coeff's:}}$ $(t^2 = 1 \cdot t^2 + 0 \cdot t + 0 \cdot 1)$

$-A = 1 \qquad A = \underline{-1}$

$-B = 0 \qquad B = \underline{0}$

$2A - C = 0 \qquad C = 2A = \underline{-2}$

$y_p = At^2 + Bt + C = -\underline{t^2 - 2}$