Plan:

1. Key problems: 10.2c, 11.16, 11.2c, 11.3bc
2. Mock exam 11/2017: 1, 3
3. Differential Equations: 1, 24, 23
4. Final exam 01/2018: 2

Key problems:

10.2 c)

$$y' = 5y\left(1 - \frac{y}{10}\right)$$

$$y' = \underbrace{y\left(1 - \frac{y}{10}\right)}_{f(y)} \cdot \underbrace{5}_{g(t)}$$

separable

$$\frac{1}{y\left(1 - \frac{y}{10}\right)} y' = 5$$

$$\underbrace{\int \frac{1}{y(1 - y/10)} \, dy}_{*} = \int 5 \, dt = 5t + C$$

$$* = \int \frac{1 \cdot 10}{y(1 - y/10) \cdot 10} dy = \int \frac{10}{y(10-y)} = \int \frac{A}{y} + \frac{B}{10-y} dy$$

$$= \int \frac{1}{y} + \frac{1}{10-y} dy = \ln|y| - \ln|10-y| + C$$

$$\frac{10}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad \cdot \, | \; y(10-y)$$

$$\boxed{\frac{10}{y(10-y)} = \frac{1}{y} + \frac{1}{10-y}}$$

$$10 = A(10-y) + By$$

$$10 = 10A + (-Ay + By)$$

$$0 \cdot y + 10 = (-A+B)y + (10A)$$

$$10A = 10 \Rightarrow A = 1$$

$$-A + B = 0 \Rightarrow B = A = 1$$

Computing both integrals, we get:

$$\ln|y| - \ln|10-y| = 5t + C \quad | \; e^{\cdot}$$

$$e^{\ln|y| - \ln|10-y|} = e^{5t+C}$$

$$\frac{|y|}{|10-y|} = e^{5t} \cdot e^{C}$$

$$\frac{y}{10-y} = \pm e^{5t} \cdot e^{C} = (\pm e^{C}) \cdot e^{5t} = K e^{5t}$$

$$\frac{y}{10-y} = K e^{5t} \quad | \cdot (10-y)$$

$$y = (10-y) \cdot K e^{5t} = 10K e^{5t} - y \cdot K e^{5t}$$

$$y + y K e^{5t} = 10 K e^{5t}$$

$$\frac{y \cdot (1 + K e^{5t})}{1 + K e^{5t}} = \frac{10 K e^{5t}}{1 + K e^{5t}}$$

$$y = \frac{10K e^{5t}}{1 + K e^{5t}}$$

11.1 b) $\underbrace{2y-3t^2}_{p}+\underbrace{2(y+t)}_{q}y'=0$

$h'_t = 2y-3t^2 \Rightarrow h=\int 2y-3t^2\,dt$
$\qquad = \underline{2yt-t^3+g(y)}$

$h'_y = 2y+2t$

Check:

$h'_y = \cancel{2t}-\cancel{0}+g'(y) = 2y+\cancel{2t}$
$g'(y)=2y$
$g(y)=y^2$

The diff. eqn. is exact and

$h = 2yt-t^3+y^2 = C$

$y^2+2t\cdot y+(-t^3-C)=0 \qquad a=1 \quad b=2t \quad c=-t^3-C$

$$y=\frac{-2t\pm\sqrt{4t^2-4\cdot1\cdot(-t^3-C)}}{2}$$

$$=-t\pm\frac{\sqrt{4t^2+4t^3+4C}}{2}=-t\pm\frac{\cancel{\sqrt{4}}\cdot\sqrt{t^2+t^3+C}}{\cancel{2}}$$

$$=\underline{\underline{-t\pm\sqrt{t^2+t^3+C}}}$$

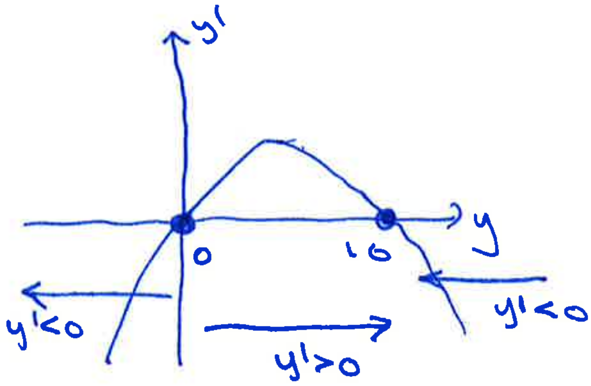
11.2 c) $y' = 5y\left(1-\frac{y}{10}\right)$

Eq. states: $y'=0$ $\quad 5y\left(1-\frac{y}{10}\right)=0$

$5y=0$, $1-\frac{y}{10}=0$

$\underline{y_e=0}$ $\quad \underline{y_e=10}$

Stability:



phase diagram

$$y' = 5y(1-y/10) = 5y - \tfrac{1}{2}y^2$$

$y_e=0$: unstable

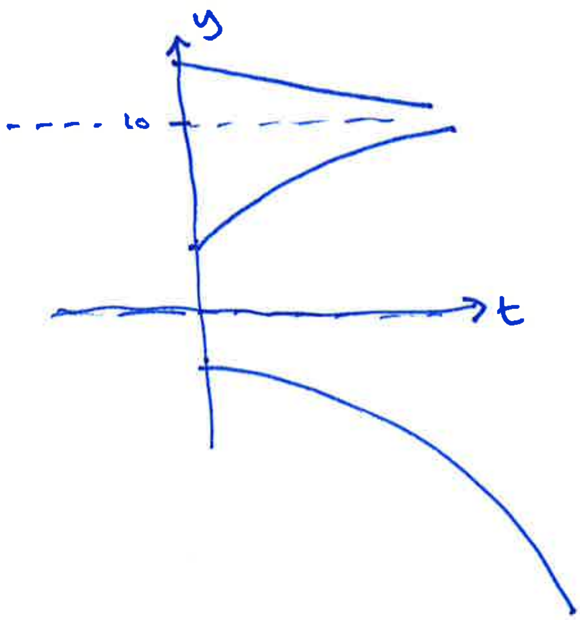
$y_e=10$: stable, not gl. as. stable

Alternative: Stability Thm.

$F'(y) = 5 - y$

$F'(0) = 5 > 0 \Rightarrow$ unstable

$F'(10) = -5 < 0$ stable



11.3 b) $y'' + 6y' + 9y = 4e^{-t}$

$y = y_h + y_p = \underline{C_1 e^{-3t} + C_2 t e^{-3t} + e^{-t}}$

y_h: $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0$

$r = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$

$r_1 = r_2 = \underline{-3} \longrightarrow y_h = \underline{C_1 \cdot e^{-3t} + C_2 t \cdot e^{-3t}}$

y_p: $y'' + 6y' + 9y = 4e^{-t}$

$\left.\begin{aligned} f &= 4e^{-t} \\ f' &= -4e^{-t} \\ f'' &= 4e^{-t} \end{aligned}\right\}$ Guess: $y = A \cdot e^{-t}$

$y' = -Ae^{-t}$

$y'' = Ae^{-t}$

$(Ae^{-t}) + 6 \cdot (-Ae^{-t})$

$+ 9(Ae^{-t}) = 4e^{-t}$

$(A - 6A + 9A)e^{-t} = 4e^{-t}$

$\underset{4}{\underset{\parallel}{(4A)}} e^{-t} = 4e^{-t}$

$4A = 4$

$\underline{A = 1}$ $\quad y_p = \underline{1 \cdot e^{-t}}$

c) $y'' - 3y' + 2y = 3e^{2t}$

$y = y_h + y_p = \underline{\underline{c_1 e^{t} + c_2 e^{2t} + 3te^{2t}}}$

$\underline{y_h}$: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 1, \ r = 2$

$y_h = \underline{c_1 \cdot e^{t} + c_2 \cdot e^{2t}}$

$\underline{y_p}$: $y'' - 3y' + 2y = 3e^{2t}$

$f = 3e^{2t}$
$f' = 6e^{2t}$
$f'' = 12e^{2t}$

$y = Ae^{2t}$
$y' = 2Ae^{2t}$
$y'' = 4Ae^{2t}$

$(4Ae^{2t}) - 3(2Ae^{2t}) + 2(Ae^{2t}) = 3e^{2t}$

$(4A - 6A + 2A)e^{2t} = 3e^{2t}$

$0 \cdot e^{2t} = 3 \cdot e^{2t}$

$\underline{\text{no solution for } A}$

$\underline{\text{Next}}$: $y = \underline{Ate^{2t}}$

$y' = A \cdot e^{2t} + At \cdot 2e^{2t}$
$= \underline{(A + 2At)e^{2t}}$

$y'' = 2A \cdot e^{2t} + (A + 2At) \cdot e^{2t} \cdot 2$
$= \underline{(4A + 4At)e^{2t}}$

$(4A + 4At)e^{2t} - 3 \cdot (A + 2At)e^{2t} + 2 \cdot (Ate^{2t}) = 3e^{2t}$

$(\underline{A} + 0 \cdot At)e^{2t} = 3 \cdot e^{2t}$

$\underset{\overset{\parallel}{3}}{(A)}\, e^{2t} = 3 \cdot e^{2t}$

$\underline{A = 3} \Rightarrow y_p = 3 \cdot \underline{te^{2t}}$

Exam	Mock exam problems in Differential Equations for GRA 6035 Mathematics
Date	November, 2017

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

Find the general solutions of the following differential equations:

(a) **(6p)** $y'' - 2y' - 3y = 4e^t$
(b) **(6p)** $2ty - 1 + t^2y' = 0$
(c) **(6p)** $2ty\,y' = 1$

Question 2.

Find all equilibrium states of the following differential equation, and determine their stability. Are any of the equilibrium states globally asymptotically stable?

(a) **(6p)** $y' = 4e^y - 2$
(b) **(6p)** $y' = y^2 - 3y + 2$

Question 3.

We consider the following system of differential equations:

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \cdot \begin{pmatrix} y \\ z \end{pmatrix}$$

(a) **(6p)** Find the general solution of the system.
(b) **(6p)** Find the equilibrium states of the system, and determine their stability.

Mock exam 11/2017:

1. c) $2ty \cdot y' = 1$

$$y' = \frac{1}{2ty} = \underbrace{\frac{1}{2y}}_{f(y)} \cdot \underbrace{\frac{1}{t}}_{g(t)} \qquad | \cdot 2y$$

$$2y \cdot y' = \frac{1}{t}$$

$$\int 2y \cdot \underbrace{y' \, dt}_{dy} = \int \frac{1}{t} \, dt$$

$$y^2 = \ln|t| + C$$

$$y = \pm\sqrt{\ln|t| + C}$$

3. $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$ $\qquad A = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix}$

$$y' = 4y - 2z$$
$$z' = -5y + z$$

General solution:

$$\begin{pmatrix} y \\ z \end{pmatrix} = c_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + c_2 \cdot \underline{v_2}\, e^{\lambda_2 t}$$

(assuming that A is diagonalizable)

Eigenvalues:

$$\begin{vmatrix} 4-\lambda & -2 \\ -5 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$\lambda^2 - 5\lambda \div 6 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot (-6)}}{2} = \frac{5 \pm 7}{2}$$

$$\underline{\lambda_1 = -1}, \quad \underline{\lambda_2 = 6}$$

E_{-1}: $\lambda = -1$

$$\begin{pmatrix} 5 & -2 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 5 & -2 & 0 \\ -5 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad 5y - 2z = 0 \Rightarrow y = \frac{2z}{5}, \quad z \text{ free}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 2/5 \cdot z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2/5 \\ 1 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} 2/5 \\ 1 \end{pmatrix}$$

E_6: $\lambda = 6$

$$\left(\begin{array}{cc|c} -2 & -2 & 0 \\ -5 & -5 & 0 \end{array}\right) \xrightarrow{-5/2} \left(\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad -2y - 2z = 0 \Rightarrow y = -z, \quad z \text{ free}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

General solution:

$$\begin{pmatrix} y \\ z \end{pmatrix} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \cdot \underline{v_2} e^{\lambda_2 t} = C_1 \cdot \begin{pmatrix} 2/5 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{6t}$$

$$= \begin{pmatrix} \frac{2}{5} C_1 e^{-t} - C_2 e^{6t} \\ C_1 e^{-t} + C_2 e^{6t} \end{pmatrix}$$

b) Eq. states: $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow A \cdot \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$|A| = (-1) \cdot 6 = -6 \neq 0$ $\qquad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

y_e globally asymptotically stable if $\lambda_1, \lambda_2 < 0$ and unstable otherwise

$y_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unstable since $\lambda_2 = 6 > 0$

Differential equations:

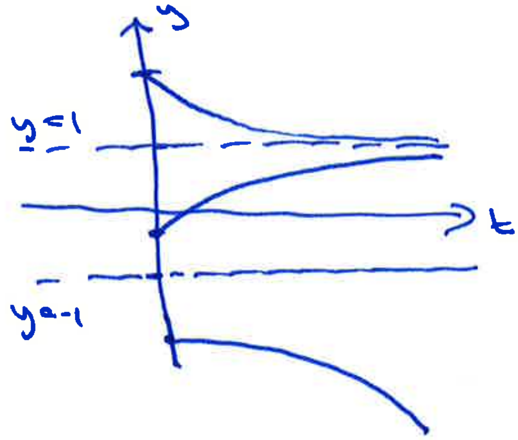
1.24. $y' = 1 - y^2$

Eq. states: $y' = 0$

$1 - y^2 = 0$ $y^2 = 1$ $y = \pm 1$

$\underline{y_e = -1}$ $\underline{y_e = 1}$

Stability:



$y' = 1 - y^2$

$y_e = 1$ is stable

(but not gl. as. stable)

$y_e = -1$ is unstable

2.3. $\underline{y}' = \underbrace{\begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}}_{A} \cdot \underline{y}$

$y_1' = 6y_1 - 3y_2$
$y_2' = -2y_1 + y_2$

$$\begin{vmatrix} 6-\lambda & -3 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 7\lambda = 0$

$\lambda(\lambda - 7) = 0 \qquad \lambda_1 = 0, \quad \lambda_2 = 7$

A diagonalizable since $\lambda_1 \neq \lambda_2$

$\underline{\lambda = 0}$: $\left(\begin{array}{cc|c} 6 & -3 & 0 \\ -2 & 1 & 0 \end{array}\right)]^{1/3} \to \left(\begin{array}{cc|c} 6 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$6y_1 - 3y_2 = 0 \Rightarrow y_1 = \frac{1}{2}y_2$
y_2 free

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y_2 \\ y_2 \end{pmatrix} = y_2 \cdot \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$\underline{v}_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$\underline{\lambda = 7}$: $\left(\begin{array}{cc|c} -1 & -3 & 0 \\ -2 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$-y_1 - 3y_2 = 0 \Rightarrow y_1 = -3y_2$
y_2 free

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -3y_2 \\ y_2 \end{pmatrix} = y_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

General solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \cdot \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} e^{0t} + C_2 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{7t} = \begin{pmatrix} \frac{1}{2}C_1 - 3C_2 e^{7t} \\ C_1 + C_2 e^{7t} \end{pmatrix}$$

Final exam 01/2018 : Question 2

a) $y'' + 2y' - 15y = 18e^{4t}$

$y = y_h + y_p = \underline{\underline{C_1 \cdot e^{-5t} + C_2 \cdot e^{3t} + 2e^{4t}}}$

y_h: $y'' + 2y' - 15y = 0$

$r^2 + 2r - 15 = 0$

$r = \dfrac{-2 \pm \sqrt{4 - 4\cdot(-15)}}{2}$

$= \dfrac{-2 \pm 8}{2} = -5, 3 \Rightarrow y_h = \underline{C_1 \cdot e^{-5t} + C_2 \cdot e^{3t}}$

y_p: $y'' + 2y' - 15y = 18e^{4t}$

$f = 18e^{4t}$
$f' = 18 \cdot 4e^{4t}$
$f'' = 18 \cdot 16 \cdot e^{4t}$

$$\boxed{\begin{aligned} y &= Ae^{4t} \\ y' &= A \cdot 4e^{4t} \\ y'' &= A \cdot 16e^{4t} \end{aligned}}$$

$(16Ae^{4t}) + 2(4Ae^{4t})$
$-15(Ae^{4t}) = 18e^{4t}$

$(16A + 8A - 15A) \cdot e^{4t} = 18e^{4t}$

$(9A)\, \underline{e^{4t}} = 18\, \underline{e^{4t}}$

$9A = 18$

$A = \underline{2} \Rightarrow y_p = \underline{2e^{4t}}$

b) $e^t \cdot y' = 2e^{y-t} \quad | \cdot e^{-t}$

$y' = 2e^{y-t} \cdot e^{-t}$

$= 2 \cdot e^y \cdot e^{-2t} = \underbrace{e^y}_{f(y)} \cdot \underbrace{2e^{-2t}}_{g(t)} \quad | \cdot e^{-y}$

$e^{-y} \cdot y' = 2e^{-2t}$

$\boxed{\begin{aligned} u &= -2t \\ du &= -2 \cdot dt \end{aligned}}$

$\int e^{-y} dy = \int 2e^{-2t} dt$

$-e^{-y} = \int \cancel{2} e^u \cdot \frac{du}{-\cancel{2}}$

$-e^{-y} = -e^u + C$

$-e^{-y} = -e^{-2t} + C \quad | (-1)$

$e^{-y} = e^{-2t} - C \quad | \ln(-)$

$-y = \ln(e^{-2t} - C)$

$y = \underline{\underline{-\ln(e^{-2t} - C)}}$

c) $y' = (y^2 - 3) \cdot \ln(y), \ y > 0$

Eq. states: $y' = 0$

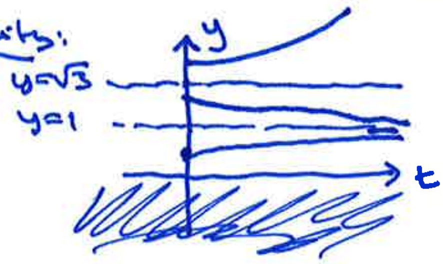
$(y^2-3) \cdot \ln(y) = 0$

$y^2 - 3 = 0$ or $\ln y = 0$

$\underline{y = \pm\sqrt{3}}$ $\quad$ $\underline{y = 1}$

$\underline{\underline{y_e = \sqrt{3}}}$, $\underline{\underline{y_e = 1}}$

Stability:



$y_e = \sqrt{3}$: unstable

$y_e = 1$: stable (but not gl. as. stable)