Plan:

Review Lecture 7

1. Second order conditions (SOC)
2. Non-degenerate constraint qualification (NDCQ)

Review: Constrained optimization

Lagrange problem:

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

$f(\underline{x}) = f(x_1, \dots, x_n)$

$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$

FOC: $\mathcal{L}'_{x_1} = 0$, $\mathcal{L}'_{x_2} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) = a_1$, $\vdots$, $g_m(\underline{x}) = a_m$

FOC + C : Lagrange Conditions

Kuhn-Tucker problem: (std. form)

max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

$f(\underline{x}) = f(x_1, \dots, x_n)$

$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$

FOC: $\mathcal{L}'_{x_1} = 0$, $\mathcal{L}'_{x_2} = 0$, $\vdots$, $\mathcal{L}'_{x_n} = 0$

C: $g_1(\underline{x}) \le a_1$, $\vdots$, $g_m(\underline{x}) \le a_m$

CSC (complementary slackness conditions):
$\lambda_1 \ge 0$ and $\lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0$
$\vdots$
$\lambda_m \ge 0$ and $\lambda_m \cdot (g_m(\underline{x}) - a_m) = 0$

FOC + C + CSC : Kuhn-Tucker conditions

Std. form: max + $g(\underline{x}) \le a$

min $f(\underline{x})$ = max $-f(\underline{x})$

$g(\underline{x}) \ge a \rightsquigarrow -g(\underline{x}) \le -a$

Ex: max/min $f(x,y) = x+3y$ when $x^2+y^2 = 10$

Lagrange

$L = x+3y - \lambda \cdot (x^2+y^2)$

FOC
$$\begin{cases} L'_x = 1 - \lambda \cdot 2x = 0 \\ L'_y = 3 - \lambda \cdot 2y = 0 \end{cases}$$
C: $x^2+y^2 = 10$

Lagrange cond.

$1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda}$ $(\lambda \neq 0)$

$3 = 2\lambda y \Rightarrow y = \frac{3}{2\lambda}$ $(\lambda \neq 0)$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$\frac{1}{(2\lambda)^2} + \frac{9}{(2\lambda)^2} = 10$

$\frac{10}{(2\lambda)^2} = 10$

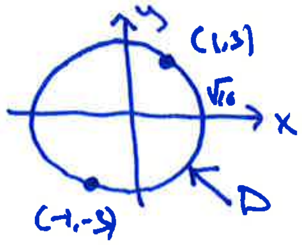
$(2\lambda)^2 = 1$ $\quad 2\lambda = \pm 1$

$\lambda = \pm 1/2$

$$\boxed{\begin{aligned} (x,y;\lambda) &= (1,3;\tfrac{1}{2}), \quad & f=10 \\ & (-1,-3;-\tfrac{1}{2}) \quad & f=-10 \end{aligned}}$$

ordinary cand. pts.

D: adm. pts



$-\sqrt{10} \le x \le \sqrt{10}$

$-\sqrt{10} \le y \le \sqrt{10}$

D is bounded

$\Downarrow$ EVT

There is a max/min

Conclude:

EVT: there is a max/min.

NDCQ: non-degenerate constraint qualification ($g = x^2+y^2 = 10$):

$J = (g'_x \;\; g'_y)$

$= (2x \;\; 2y)$

Jacobian matrix of g

NDCQ: $\text{rk } J = 1$

NDCQ fails: $\text{rk } J < 1$

$2x = 2y = 0$

not adm. $\rightarrow (x,y) = (0,0)$

not adm. pts where NDCQ fails

Thm

If $\underline{x}^*$ is max/min in a Lagrange problem (max in a Kuhn-Tucker problem in std. form), and NDCQ holds at $\underline{x}^*$, then there are Lagrange multipliers $\underline{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*)$ such that

$(\underline{x}^*; \lambda^*)$ satisfies FOC+C (FOC+C+CSC)

↑ Lagrange case ↑ Kuhn-Tucker case

Candidate points:

1) Solutions of FOC+C / FOC+C+CSC ← ordinary cand. pts
2) Admissible pts where NDCQ fails ← exceptions

Conclusions about max/min:

1) Second order condition (SOC)
2) Extreme value theorem (EVT)

Extreme value theorem:

Let D be the set of all admissible points (all points that satisfy all constraints).

If D is bounded, then f has a max and a min.

Method:

1. Find all ordinary candidate pts (solve FOC+C / FOC+C+CSC)
2. Check if there are exceptional cand. pts where NDCQ fails
3. Conclude using SOC / EVT or some other method.

Conclusion: $f_{max} = 10$ at $(x,y) = (1,3)$ with $\lambda = 1/2$
$f_{min} = -10$ at $(-1,-3)$, $\lambda = -1/2$

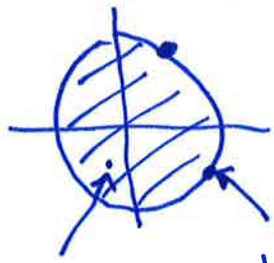
Ex: max $x+3y$ when $x^2+y^2 \le 10$ (KT std. form)

$\mathcal{L} = x+3y - \lambda(x^2+y^2)$

FOC: $\begin{cases} L'_x = 1 - \lambda \cdot 2x = 0 \\ L'_y = 3 - \lambda \cdot 2y = 0 \end{cases}$

C: $x^2+y^2 \le 10$

CSC: $\lambda \ge 0$
$\lambda \cdot (x^2+y^2-10) = 0$



$(x,y;\lambda) = (1,3;1/2)$ $f=10$ satisfies FOC+C+CSC.

Second order condition: $h(x,y) = \mathcal{L}(x,y;1/2)$
$= x+3y - \frac{1}{2}(x^2+y^2)$

SOC: h concave $\Rightarrow$ $(x,y)=(1,3)$ is max

$h'_x = 1-x$
$h'_y = 3-y$

$H(h) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $D_1 = -1$, $D_2 = 1$

neg. defn.
$\Downarrow$
h concave
$\Downarrow$
Concl: $(x,y) = (1,3)$ is max $f_{max} = 10$

(1) Second order condition:

Thm: Second order condition (SOC)

Let us consider a Lagrange problem or a Kuhn-Tucker problem in std. form with Lagrangian $\mathcal{L}$.

If $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC+C / FOC+C+CSC, then we have:

$$\boxed{\begin{array}{l} h(\underline{x}) \text{ convex} \Rightarrow \underline{x}^* \text{ is } \underline{\text{min}} \\ h(\underline{x}) \text{ concave} \Rightarrow \underline{x}^* \text{ is } \underline{\text{max}} \end{array}}$$

where $h(\underline{x}) = \mathcal{L}(\underline{x}; \underline{\lambda}^*) = \mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$

Ex: $\min f(x,y,z) = 2x^2 + y^2 + 3z^2$ when $\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$

$\boxed{\text{KT std. form}} = \max -f(x,y,z) = -2x^2 - y^2 - 3z^2$ when $\begin{cases} -x + y - 2z \leq -3 \\ -x - y \leq -3 \end{cases}$

$$\mathcal{L} = -2x^2 - y^2 - 3z^2 - \lambda_1(-x + y - 2z) - \lambda_2(-x - y)$$

FOC
$$\begin{cases} \mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0 \\ \mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0 \\ \mathcal{L}'_z = -6z + 2\lambda_1 = 0 \end{cases}$$

C
$$\begin{cases} x - y + 2z \geq 3 \\ x + y \geq 3 \end{cases}$$

CSC
$$\begin{cases} \lambda_1 \geq 0, \ \lambda_1(x - y + 2z - 3) = 0 \\ \lambda_2 \geq 0, \ \lambda_2 \cdot (x + y - 3) = 0 \end{cases}$$

<u>Cases:</u>

A: $g_1 = 3$, $g_2 = 3$ $\quad \lambda_1, \lambda_2 \geq 0$

B: $g_1 = 3$, $g_2 > 3$ $\quad \lambda_1 \geq 0, \lambda_2 = 0$

C: $g_1 > 3$, $g_2 = 3$ $\quad \lambda_1 = 0, \lambda_2 \geq 0$

D: $g_1 > 3$, $g_2 > 3$ $\quad \lambda_1 = \lambda_2 = 0$

A:

FOC + C + CSC

$$\boxed{\begin{aligned} x-y+2z&=3 \\ x+y&=3 \\ -4x+\lambda_1+\lambda_2&=0 \\ -2y-\lambda_1+\lambda_2&=0 \\ -6z+2\lambda_1&=0 \end{aligned}}$$

$\lambda_1 \geq 0$
$\lambda_2 \geq 0$

Note that this is a 5×5 lin. system in $x, y, z, \lambda_1, \lambda_2$; it can alternatively be solved using Gauss elimination

$4x = \lambda_1+\lambda_2$ $\quad x = \frac{1}{4}(\lambda_1+\lambda_2)$

$2y = -\lambda_1+\lambda_2$ $\quad y = \frac{1}{2}(-\lambda_1+\lambda_2)$

$6z = 2\lambda_1$ $\quad z = \frac{1}{6}(2\lambda_1)$

$$\frac{1}{4}(\lambda_1+\lambda_2) - \frac{1}{2}(-\lambda_1+\lambda_2) + \frac{2}{6}(2\lambda_1) = 3 \qquad 1\cdot 12$$

$$3(\lambda_1+\lambda_2) - 6(-\lambda_1+\lambda_2) + 4\cdot(2\lambda_1) = 36$$

I $\boxed{17\lambda_1 - 3\lambda_2 = 36}$

$$\frac{1}{4}(\lambda_1+\lambda_2) - \frac{1}{2}(-\lambda_1+\lambda_2) = 3 \qquad 1\cdot 4$$

$$\lambda_1+\lambda_2 + 2(-\lambda_1+\lambda_2) = 12$$

II $\boxed{-\lambda_1 + 3\lambda_2 = 12}$

$-3+3\lambda_2 = 12$

$\frac{3\lambda_2}{3} = \frac{15}{3}$

I+II: $\frac{16\lambda_1}{16} = \frac{48}{16}$ $\quad \underline{\lambda_1 = 3}$ $\quad \underline{\lambda_2 = 5}$

$x=2 \quad y=1 \quad z=1$

A: one candidate pt : $(x,y,z;\lambda_1,\lambda_2) = \underline{(2,1,1;3,5)}$

$$h(x,y,z) = -2x^2 - y^2 - 3z^2 - 3(-x+y-2z) - 5(-x-y)$$

$$H(h) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

neg. defn $\Rightarrow$ h concave

$\Rightarrow (2,1,1)$ max for $-f$

$\underline{\underline{(2,1,1) \text{ min for } f}}$

$D_1 = -4$
$D_2 = 8$
$D_3 = -48$

$-f = -12$ max value for $-f$

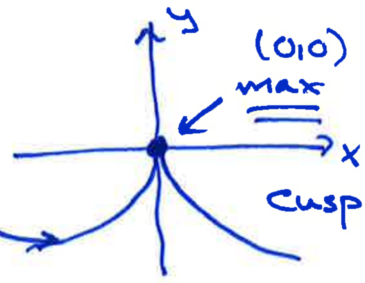
$f = 12$ min. value for f

(2) NDCQ condition:

$y^3 = -x^2$

$y = \sqrt[3]{-x^2} = -\sqrt[3]{x^2}$

Ex: max y when $x^2+y^3=0$



$L = y - \lambda\cdot(x^2+y^3)$

$L'_x = -\lambda\cdot 2x = 0$ $\quad \cancel{\lambda = 0}$ or $\cancel{x=0}$

$L'_y = 1 - \lambda\cdot 3y^2 = 0$ $\quad \cancel{1=0}$ $\quad \cancel{1=0}$

$x^2+y^3 = 0$ $\quad \cancel{y^3=0}$ $\quad \cancel{y=0}$

no ordinary cand. pts. (FOC+C)

NDCQ: $g = x^2+y^3=0$

$J = (2x \quad 3y^2) \leftarrow J = (g'_x \quad g'_y)$

NDCQ: $\text{rk } J = 1$

NDCQ fails: $\text{rk } J < 1$

$2x = 3y^2 = 0$

$(x,y) = (0,0)$ adm. pt! $\Rightarrow$ cand. for max

NDCQ in the Lagrange case:

$$\left.\begin{aligned} g_1(\underline{x}) &= a_1 \\ g_2(\underline{x}) &= a_2 \\ &\vdots \\ g_m(\underline{x}) &= a_m \end{aligned}\right\} \qquad J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 & \cdots & \partial g_2/\partial x_n \\ \vdots & & & \vdots \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

$\underline{x} = (x_1, \ldots, x_n)$

Jacobian matrix

NDCQ: $\operatorname{rk} J = m$ $\quad$ (m = # constraints)

NDCQ fails: $\operatorname{rk} J < m \iff$ all m-minors are zero

Ex:

$$\begin{aligned} x - y + 2z &= 3 \\ x + y \phantom{{}+2z} &= 3 \end{aligned} \qquad J = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

NDCQ: $\boxed{\operatorname{rk} J = 2}$

$M_{12,12} = 1 + 1 = 2 \neq 0$

$\Rightarrow \operatorname{rk} J = 2$

$\boxed{\text{NDCQ satisfied at all adm. points}}$

Kuhn-tucker case:

$$\left.\begin{array}{l} g_1(x_1,\dots,x_n) \le a_1 \\ \vdots \\ g_m(x_1,\dots,x_n) \le a_m \end{array}\right\}$$

$$J = \begin{pmatrix} \partial g_1/\partial x_1, & \partial g_1/\partial x_2 & \cdots \\ \vdots & & \end{pmatrix}$$

(same as in the Lagrange case)

$(x_1^*,\dots,x_n^*)$ adm pt:

S be all indices i such that $g_i(\underline{x}^*) = a_i$

J_S = submatrix of J consisting of indices in S $\Leftrightarrow$ binding constraints

NDCQ at $\underline{x}^*$: $\operatorname{rk} J_S$ = # binding constraints

Ex: $x - y + 2z \ge 3$
$x + y \ge 3$

$$J = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

A) $g_1 = 3,\ g_2 = 3$: NDCQ: $\operatorname{rk} J = 2$ ✓ok.

B) $g_1 = 3,\ g_2 > 3$: $\operatorname{rk}\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = 1$ ✓ok.

c) $g_1 > 3,\ g_2 = 3$: $\operatorname{rk}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = 1$ ✓ok.

d) $g_1 > 3,\ g_2 > 3$: no condition ✓ok.

For each point, use the matrix obtained by including rows corresponding binding constraints. The rank should be maximal, ie equal #rows.