Plan:

Review Lecture 4

① Diagonalization and Markov chains

② Quadratic forms and definiteness

Review: Lecture 4

A
$n \times n$-matrix

$$\boxed{A \cdot \underline{v} = \lambda \cdot \underline{v}}$$
$$\Updownarrow$$
$$(A - \lambda I)\underline{v} = \underline{0}$$

Eigenvalues:

Solve $\det(A - \lambda I) = 0$

n'th order poly. eqn.

Note: $\boxed{\text{multiplicity of } \lambda}$

Eigenvectors:

$E_\lambda = \text{Null}(A - \lambda I)$: $\left\{ \begin{array}{l} \text{Solutions of} \\ (A - \lambda I)\underline{v} = \underline{0} \end{array} \right.$

dim E_λ = # free var's in ↑

Note: λ eigenvalue w/ mult. m

$$\Downarrow$$

$$\boxed{\begin{array}{c} 1 \leq \dim E_\lambda \leq m \\ \shortparallel \\ \#\text{free var's} \end{array}}$$

Diagonalization:

A is diagonalizable if there is an invertible matrix P s.t.

$$P^{-1} A P = D$$

where D is diagonal.

Criterion:

A diagonalizable ⟺ i) A has n eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity), and

ii) A has n linearly independent eigenvectors $\underline{v_1}, \dots, \underline{v_n}$.

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \longleftrightarrow \text{i)}$$

$$\boxed{A \cdot v_i = \lambda_i \cdot v_i}$$

$$P = \left(\underline{v_1} \middle| \underline{v_2} \middle| \cdots \middle| \underline{v_n} \right) \longleftrightarrow \text{ii)}$$

Result:

A symmetric ⟹ A diagonalizable

A has n eigenvalues of multiplicity 1 } ⟹ A diagonalizable

Note: these implications go in one direction only!

Facts: $\boxed{\text{ii)} \Longleftrightarrow \text{For each eigenvalue of mult. } m, \ \dim E_\lambda = m.}$

↑

Automatically satisfied if m=1!

① Markov chains

Ex: unemployment, US weekly data

e_t = share of pop. that is employed in week t

u_t = — " — unemployed — " —

If unemployment is 10%:

$e_0 = 0.90$

$u_0 = 0.10$

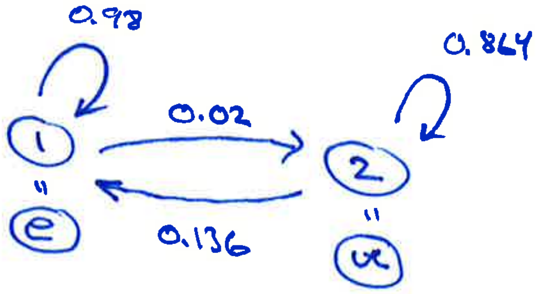
t=0: $\underline{x}_0 = \begin{pmatrix} e_0 \\ u_0 \end{pmatrix}$

initial state

general: $\underline{x}_t = \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

state vector

$$\boxed{\begin{array}{l} e_t \geq 0, \; u_t \geq 0 \\ e_t + u_t = 1 \end{array}}$$



$$A = \begin{array}{ccc} e & u & \\ \left(\begin{array}{c} 0.98 \\ 0.02 \end{array}\right. & \left.\begin{array}{c} 0.136 \\ 0.864 \end{array}\right) & \begin{array}{c} e \\ u \end{array} \end{array}$$

transition matrix

$$\begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$$

$$\boxed{\begin{array}{l} a_{ij} \geq 0 \quad \text{for all } ij \\ \text{each col. sum is } 1. \end{array}}$$

$\underline{x}_1 = A \cdot \underline{x}_0$

$$\begin{pmatrix} e_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ u_0 \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} \overset{ee}{0.98 \cdot e_0} + \overset{ue}{0.136 \cdot u_0} \\ \underset{eu}{0.02\, e_0} + \underset{uu}{0.864 \cdot u_0} \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{Markov chain:} \\ \text{constant transition matrix} \end{array}}$$

$\underline{x}_1 = A \cdot \underline{x}_0$

$\underline{x}_2 = A \cdot \underline{x}_1 = A \cdot (A \underline{x}_0) = A^2 \cdot \underline{x}_0$

$$= \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ u_0 \end{pmatrix}$$

$$\vdots \quad = \begin{pmatrix} 0.98 \cdot 0.98 + 0.136 \cdot 0.02 & * \\ 0.02 \cdot 0.98 + 0.864 \cdot 0.02 & * \end{pmatrix} \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \overset{eee}{0.98 \cdot 0.98 \cdot e_0} + \overset{eue}{0.136 \cdot 0.02 \cdot e_0} \; \cdots \\ \cdots \end{pmatrix}$$

$\underline{x}_n = A^n \cdot \underline{x}_0$

$\lim_{n\to\infty} \underline{x}_n = \lim_{n\to\infty} A^n \cdot \underline{x}_0$ ← Equilibrium state (if the limit exists)

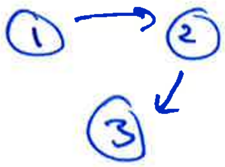
How to compute "long run"? $\boxed{A^n \cdot \underline{x}_0}$ for n big

* compute A^n using a diagonalization of A (if it exists)
* using theory for Markov chain

Defn: A Markov chain is called regular if A^n consists of non-zero entries for n big enough.

$$\boxed{\text{Ex: } A = \begin{pmatrix} 0.98 & 0.156 \\ 0.02 & 0.864 \end{pmatrix} \quad \text{regular}}$$

is strongly connected



$a_{13}=0$ but

$A^2_{13} \neq 0$

Thm:
If A is the transition matrix of a regular Markov chain, then:

i) there is a unique eigenvector $\underline{v}$ of A with eigenvalue $\lambda = 1$ that is a state vector. Recall: state vector $\underline{v}$ means that $v_i \geq 0$ for all i, $v_1+v_2+\dots+v_n=1$.

$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

ii) for any initial state $\underline{x}_0$, we have that

$$\lim_{n\to\infty} A^n \underline{x}_0 = \underline{v}$$

and $\underline{v}$ is called the equilibrium state. Moreover, we have that

$$\lim_{n\to\infty} A^n = (\underline{v}|\underline{v}|\dots|\underline{v})$$

Ex: $A = \begin{pmatrix} 0.98 & 0.136 \\ 0.02 & 0.864 \end{pmatrix}$ regular ✓ $a_{11} > 0 \quad a_{12} > 0$ $a_{21} > 0 \quad a_{22} > 0$

$$\begin{vmatrix} 0.98 - 1 & 0.136 \\ 0.02 & 0.864 - 1 \end{vmatrix} = 0$$

E_1: $\begin{pmatrix} \boxed{-0.02} & 0.136 \\ 0.02 & -0.136 \end{pmatrix}$ $-0.02x + 0.136y = 0$

y free

$\Downarrow$

$$\frac{0.02x}{0.02} = \frac{0.136y}{0.02}$$

$$x = \frac{0.136}{0.02} y = \frac{13.6}{2} y = 6.8y$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.8y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 6.8 \\ 1 \end{pmatrix}$$

State vectors:

$y \geq 0 \quad 6.8y + y = 1$

$7.8y = 1$

$y = \frac{1}{7.8}$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.8 \cdot \frac{1}{7.8} \\ \frac{1}{7.8} \end{pmatrix} = \begin{pmatrix} 6.8/7.8 \\ 1/7.8 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix} \leftarrow \begin{pmatrix} e \\ u \end{pmatrix}$$

Long run unemployment: $10/78 \simeq \underline{\underline{12.8\%}}$

Alternative computations:

i) Use Wolfram Alpha to compute A^n for n big enough.

ii) Use diagonalization:

Find eigenvalues of A: $\xrightarrow{\text{skipping some details}}$ $\lambda = 1, \ \lambda = 0.864$

Find eigenvectors of A: $\longrightarrow$ $\underline{v}_1 = \begin{pmatrix} 68/78 \\ 10/78 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

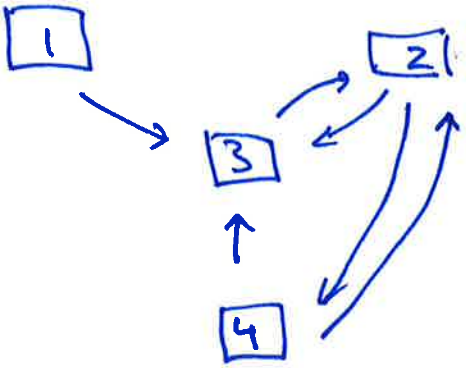
$P^{-1}AP = D$

$A = PDP^{-1}$

Compute $A^n = PD^nP^{-1} = \begin{pmatrix} 68/78 & 1 \\ 10/78 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.864^n \end{pmatrix} \cdot \frac{1}{(-1)} \cdot \begin{pmatrix} -1 & -1 \\ -10/78 & 68/78 \end{pmatrix}$

$\boxed{n \to \infty}$ $\longrightarrow \begin{pmatrix} 68/78 & 1 \\ 10/78 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 10/78 & 68/78 \end{pmatrix} = \begin{pmatrix} 68/78 & 68/78 \\ 10/78 & 10/78 \end{pmatrix} = (\underline{v} | \underline{v})$

Ex: Google Pagerank algorithm



$$M = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 1/2 & 2 \\ & 1 & 1/2 & 0 & 1/2 & 3 \\ & 0 & 1/2 & 0 & 0 & 4 \end{array}$$

if there is a path from node j to node i, let $m_{ij} = \frac{1}{n}$ where n = # paths starting from node j

Equilibrium state:

$$M^n \cong \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.44 & 0.44 & 0.44 & 0.44 \\ 0.33 & 0.33 & 0.33 & 0.33 \\ 0.22 & 0.22 & 0.22 & 0.22 \end{pmatrix}$$

for n large (using Wolfram Alpha)

Note: this Markov chain is not regular

no path from node 3 to node 1, for example, not strongly connected graph. Still, we have:

Interpretation:

Rating of web pages

2 : 4/9 ← "Best" web page = first ranking.

3 : 3/9

4 : 2/9

1 : 0

$\lambda = 1$:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1/2 \\ 1 & 1/2 & -1 & 1/2 \\ 0 & 1/2 & 0 & -1 \end{pmatrix} \to \ldots \to \begin{pmatrix} \enclose{circle}{-1} & 0 & 0 & 0 \\ 0 & \enclose{circle}{-1} & 1 & 1/2 \\ 0 & 0 & \enclose{circle}{-1/2} & 3/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_4 free

E_1: $-\frac{1}{2}x_3 + \frac{3}{4}x_4 = 0 \Rightarrow x_3 = \frac{3}{2}x_4$

$-x_2 + x_3 + \frac{1}{2}x_4 = 0 \qquad x_2 = \frac{3}{2}x_4 + \frac{1}{2}x_4 = 2x_4$

$-x_1 = 0 \qquad x_1 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 0 \\ 2 \\ 3/2 \\ 1 \end{pmatrix} \Rightarrow \underline{v} = \frac{2}{9}\begin{pmatrix} 0 \\ 2 \\ 3/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4/9 \\ 3/9 \\ 2/9 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0.44 \\ 0.33 \\ 0.22 \end{pmatrix}$$

state vector:

$0 + 2x_4 + \frac{3}{2}x_4$

$+ x_4 = 1$

$\Rightarrow \frac{9}{2}x_4 = 1$

$x_4 = \frac{2}{9}$

② Quadratic forms and definiteness

Ex: $f(x,y,z) = x^2 - 2yz$ is a quadratic form

Defn: $f(x_1,x_2,\ldots,x_n)$ is a quadratic form if it is a polynomial where each term has degree two

n=1: $f(x) = ax^2$

n=2: $f(x,y) = ax^2 + bxy + cy^2$

n=3:
$$f(x,y,z) = ax^2 + bxy + cxz + dy^2 + eyz + f\cdot z^2$$

Result: Any quadratic form can be written as
$$f(x_1,\ldots,x_n) = \underline{x}^T A \underline{x} \quad \text{with} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$
for a unique symmetric $\underset{n\times n}{\wedge}$ matrix A.

Ex:

$f(x,y,z) = x^2 - 2yz$

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (x \;\; y \;\; z)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x^2 - 2yz)$$

$$= \begin{pmatrix} x & -z & -y \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x^2 - zy - yz)$$
$$\|$$
$$(x^2 - 2yz)$$

Note: $f(x_1,\dots,x_n)$ quadratic form

i) $f(0,0,\dots,0) = 0$

ii) $(0,0,\dots,0)$ is a stationary pt.

Ex: $f = x^2 - 2yz$

$f'_x = 2x$

$f'_y = -2z$

$f'_z = -2y$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{x} = (x_1, x_2, \dots, x_n)$

Defn:

f is called

- positive semidefinite if $f(\underline{x}) \geq 0$ for all $\underline{x} = x_1,\dots,x_n$
- negative semidefinite if $f(\underline{x}) \leq 0$ —"—
- indefinite otherwise (f takes both pos. and neg. values)

- positive definite if $f(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$
- negative definite if $f(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$

When we write $f(x_1,\dots,x_n) = f(\underline{x}) = \underline{x}^T A \underline{x}$ for an unique symmetric $n \times n$ quadratic form -matrix A, we can use matrix methods to determine the definiteness of A/f.

Result: If A has eigenvalues $\lambda_1,\dots,\lambda_n$, then:

A pos. defn. $\Longleftrightarrow$ $\lambda_1,\dots,\lambda_n > 0$

A pos. semidefn. $\Longleftrightarrow$ $\lambda_1,\dots,\lambda_n \geq 0$

A neg. defn. $\Longleftrightarrow$ $\lambda_1,\dots,\lambda_n < 0$

A neg. semidefn. $\Longleftrightarrow$ $\lambda_1,\dots,\lambda_n \leq 0$

A indefinite $\Longleftrightarrow$ there are both pos. and neg. λ.

Ex: $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ x^2+3y^2

pos. defn $\lambda=1, \lambda=3$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ x^2-3y^2

$\lambda=1, \lambda=-3$

indefn.

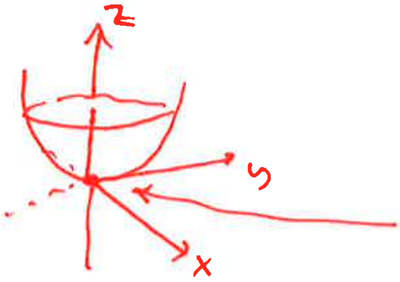
$A = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$ $-x^2-3y^2$

neg. defn.

$\lambda=-1$ $\lambda=-3$

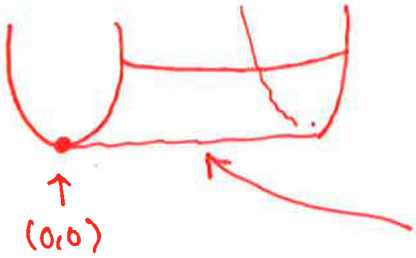
Definiteness: Interpretation in terms of the graph of the quadratic form f

f pos. defn. :



global min : (0,0)

f pos. semidefn. :

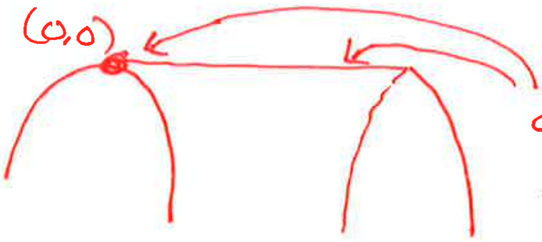


line of global min's including (0,0)

f neg. defn :

global max (0,0)

f neg. semidefn:



a line of global max incl. (0,0)

f indefinite : (0,0) is saddle point.

Can be difficult to compute eigenvalues. Alternative methods using principal minors, which are easy to compute.

Principal minors:

A minor is principal if we keep certain rows $i_1, i_2, \dots, i_r$ and keep cols $i_1, i_2, \dots, i_r$.

A leading principal minor: We keep rows/cols $1, 2, \dots, r$.

Ex: $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & 3 & 1 \end{pmatrix}$ indefinite

$$f(x,y,z) = x^2 + 4y^2 + z^2 + 4xy - 2xz + 6yz$$

Leading principal minors:

$D_1 = |1| = 1$

$D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$D_3 = |A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & 3 & 1 \end{vmatrix} =$

$= 1 \cdot (-5) - 2 \cdot 5 - 1 \cdot 10$

$= \underline{-25}$

Principal minors:

$\Delta_1 = 1, 4, 1$

$\Delta_2 = \underset{M_{12,12}}{\underset{\shortparallel}{0}}, \underset{M_{23,23}}{\underset{\shortparallel}{-5}}, \underset{M_{13,13}}{\underset{\shortparallel}{0}}$

$\Delta_3 = \underset{|A|}{\underset{\shortparallel}{-25}}$

Result: If A is an $n \times n$ symmetric matrix, then

i) A pos. defn. $\Longleftrightarrow D_1, D_2, \dots, D_n > 0$

ii) A neg. defn. $\Longleftrightarrow D_1 < 0, D_2 > 0, \dots \Longleftrightarrow (-1)^i D_i > 0$

iii) A pos. semidefn $\Longleftrightarrow \Delta_1, \Delta_2, \dots, \Delta_n \geq 0 \leftarrow$ for all principal minors Δ_i

iv) A neg. semidefn $\Longleftrightarrow \Delta_1 \leq 0, \Delta_2 \geq 0, \dots \Longleftrightarrow (-1)^i \Delta_i \geq 0$

v) A indefinite $\Longleftrightarrow$ neither positive semidefn nor neg. semidefn.

Ex: $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

neg. defn.

$D_1 = -1$

$D_2 = (-1)\cdot(-2) = 2$

$D_3 = (-1)\cdot(-2)(-3) = -6$

$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$

alternating pattern when A is neg. defn! →

$$\begin{cases} D_1 = \lambda_1 = -1 \\ D_2 = \lambda_1\lambda_2 = 2 \\ D_3 = \lambda_1\lambda_2\lambda_3 = -6 \end{cases}$$

Ex: $f = x^2 + y^2 + z^2 - 2xz \iff A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

Leading principal minors:

$D_1 = 1$

$D_2 = 1 \leftarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$D_3 = 0 \leftarrow |A| = 1 \cdot (1 - (-1)^2) = 0$

$\Delta_1 = 1, 1, 1$

$\Delta_2 = 1, 1, 0$

$\Delta_3 = 0$

- can be pos. semidefn. (all $\Delta_i \geq 0$) or indefinite.
- check all principal minors →

all principal minors are ≥ 0

$\Updownarrow$

f positive semi. defn.

Ex: $f = x^2 + 4xy + 8xz + 3y^2 - 2yz + 2z^2 \iff A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$

$D_1 = 1$

$D_2 = 3 - 4 = -1 \leftarrow$ indefinite

$$\begin{cases} \text{pos. semidefn: all } \Delta_2 \geq 0 & \text{No!} \\ \text{neg. —''— all } \Delta_2 \geq 0 & \text{No!} \end{cases}$$

$\Updownarrow$

indefinite