Plan:

Review Lecture 11

1. Systems of differential equations
2. Difference equations

Lecture 13: Final exam 11/2017.
Lecture 14: Review / TBA.

Monday: Plenary Session – Selected problems from Lecture 10–12.

Review:

First order differential equations

(a) Separable: Method: Separation of variables

$y' = f(y) \cdot g(t)$

$\frac{1}{f(y)} \cdot y' = g(t)$

$$\boxed{\int \frac{1}{f(y)}\, dy = \int g(t)\, dt}$$

(b) Linear: Method: Integrating factor $u = e^{\int a(t)\, dt}$

$y' + a(t) \cdot y = b(t)$

$(y \cdot u)' = b(t) \cdot u(t) \Rightarrow \boxed{y = \frac{1}{u} \int b(t)\, u(t)\, dt}$

(c) Exact: Method: Partial integration

$p(t,y) + q(t,y) \cdot y' = 0$ with $h'_t = p$ and $h'_y = q$ for some fn. $h(t,y)$

$\boxed{h(t,y) = C}$

$\Uparrow$ check exactness

$p'_y = q'_t$

Second order differential equations

(d) Linear with constant coefficients

$y'' + ay' + by = f(t)$

Can also be used to solve $y' + ay = b(t)$

Method: superposition

$y = y_h + y_p$

y_h: General solution of $\underbrace{y'' + ay' + by = 0}_{\text{homogeneous}}$

Char. eqn: $r^2 + ar + b = 0$, $r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

$r_1 \neq r_2$: $y_h = \boxed{C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}}$

$r_1 = r_2$: $y_h = \boxed{C_1 e^{r_1 t} + C_2 t e^{r_2 t}}$

y_p: Particular solution of $\underbrace{y'' + ay' + by = f(t)}_{\text{inhomogeneous}}$

Method of undetermined coeff's:

Guess a form of $y(t)$ based on $f(t), f'(t), f''(t)$ with undetermined coeff's, substitute $y(t)$ in the diff. eqn.

If there are no solutions, try with $t \cdot y(t)$.

Ex: $y e^{ty} + t e^{ty} \cdot y' = 0$, $y(1) = \ln(2)$

$p = y e^{ty} = h'_t$ $\quad h = \int y e^{ty} \, dt =$ $\boxed{\begin{array}{l} u = ty \\ du = y \cdot dt \end{array}}$

$q = t e^{ty} = h'_y$

$h'_y = \boxed{\cancel{e^{ty} \cdot t} + g'(y) = \cancel{t e^{ty}}}$ $\quad = \int \cancel{y} \cdot e^u \cdot \frac{du}{\cancel{y}} = \int e^u \, du$

Can choose $g(y) = 0$. $\quad h = e^u + g(y) = e^{ty} + g(y)$

exact !!

$h = e^{ty} = C$

$\dfrac{ty}{t} = \dfrac{\ln(C)}{t}$

$y = \dfrac{\ln(C)}{t}$

$y(1) = \ln(2)$

$\ln(2) = \dfrac{\ln(C)}{1}$

$C = 2$

$y = \dfrac{\ln(2)}{t}$

Ex: $y'' - 29y' + 100y = 100t - 29$

$y = y_h + y_p = \underline{C_1 e^{4t} + C_2 e^{25t} + t}$

y_h: $y'' - 29y' + 100y = 0$

$r^2 - 29r + 100 = 0$

$r_1 = 4, \ r_2 = 25 \implies y_h = \underline{C_1 e^{4t} + C_2 e^{25t}}$

y_p: $y'' - 29y' + 100y = 100t - 29$

$f(t) = 100t - 29$
$f' = 100$
$f'' = 0$

⇓ guess

$y = \underline{At + B}$
$y' = A$
$y'' = 0$

$0 - 29 \cdot A + 100(At + B) = 100t - 29$

$(100A) \cdot t + (-29A + 100B) = 100t - 29$

$100A = 100$
$-29A + 100B = -29$

$A = 1$
$100B = -29 + 29 = 0$
$B = 0$

$\underline{y_p = t}$

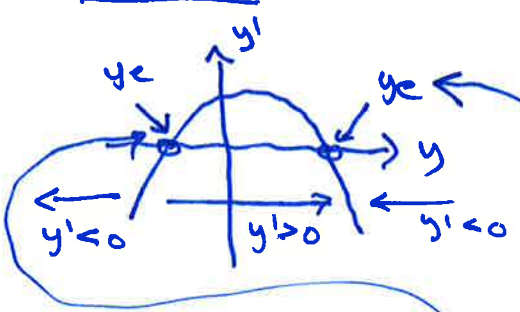
Equilibrium states and stability:

$y' = F(y)$ autonomous case

Eq. state: $y' = 0 \iff \boxed{F(y) = 0}$
(= constant solution)

Stability: y_e eq. state

Stability Thm:
$F'(y_e) > 0 \implies y_e$ unstable
$F'(y_e) < 0 \implies y_e$ stable



phase diagram

y_e is not globally asymptotically stable

$y_0 < y_e$ near, $y(t)$ moves away

Ex: $y' = y^3 - y$

Eq. states: $y^3 - y = 0$

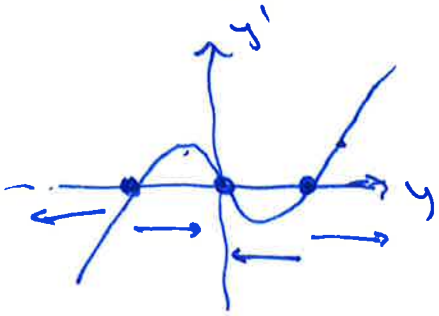
$y(y^2-1) = 0$

$y = 0$ or $y^2 = 1$

$y = \pm 1$

$y_e = 0, \; y_e = 1, \; y_e = -1$

Stability:



phase diagram

$y_e = \pm 1$: unstable

$y_e = 0$: stable

not gl. as. stable since

$y_0 > 1 \Rightarrow y(t)$ moves away from $y_e = 0$ as $t \to \infty$

$y_0 < -1$

① Systems of differential equations

Ex: $y_1' = y_1 - 2y_2$, $y_2' = -2y_1 + y_2$ in $y_1(t), y_2(t)$ "coupled"

In matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Leftrightarrow \underline{y}' = A \cdot \underline{y}, \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\underline{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

Diagonal case:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Leftrightarrow \begin{array}{l} y_1' = 3y_1 \\ y_2' = -y_2 \end{array}$$

$$\Downarrow$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} \\ C_2 e^{-t} \end{pmatrix} \Leftarrow \begin{cases} y_1 = C_1 \cdot e^{3t} \\ y_2 = C_2 \cdot e^{-t} \end{cases}$$

linear or ~~separable~~ separable

$y_1' = 3y_1 \Rightarrow y_1' - 3y_1 = 0$

$r - 3 = 0 \Rightarrow r = 3, \; y_1 = C_1 e^{3t}$

General linear system of differential equations:

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_n' \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$ where A is $n \times n$-matrix.

Solution method: Diagonalizing A

If A is diagonalizable, with eigenvalues $\lambda_1, \dots, \lambda_n$ and linearly independent eigenvectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$, such that $A \cdot \underline{v_i} = \lambda_i \cdot \underline{v_i}$, then the general solution of $\underline{y}' = A \cdot \underline{y}$ is given by:

$$\boxed{\underline{y} = C_1 \cdot \underline{v_1} \cdot e^{\lambda_1 t} + C_2 \cdot \underline{v_2}\, e^{\lambda_2 t} + \dots + C_n \cdot \underline{v_n} \cdot e^{\lambda_n t}}$$

Ex:
$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= -2y_1 + y_2 \end{aligned} \iff \underline{y}' = A \cdot \underline{y} \qquad A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

Eigenvalues:
$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda - 3 = 0$$

$$(1-\lambda)^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2}$$

$$\lambda_1 = \underline{3}, \quad \lambda_2 = \underline{-1}$$

Eigenvectors:

E_3: $\begin{pmatrix} -2 & -2 \\ \cancel{-2} & \cancel{-2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\left.\begin{matrix} x = -y \\ y \text{ free} \end{matrix}\right\}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = \underline{y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$ $\quad \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{E_{-1}}$: $\begin{pmatrix} 2 & -2 \\ \cancel{-2} & \cancel{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\left.\begin{matrix} x = y \\ y \text{ free} \end{matrix}\right\}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \underline{y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$: $\lambda_1 = 3$, $\lambda_2 = -1$

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Downarrow$

$\underline{y}' = A\underline{y}$

$$\underline{y} = C_1 \cdot v_1 \cdot e^{\lambda_1 t} + C_2 \cdot \underline{v_2}\, e^{\lambda_2 t}$$

$$= c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -c_1 e^{3t} + c_2 e^{-t} \\ c_1 e^{3t} + c_2 e^{-t} \end{pmatrix}$$

__Explanation:__ $\underline{y}' = A \cdot \underline{y}$

Assume A is __diagonalizable__ with

$$\left\{\begin{array}{l} D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \\ P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right) \\ P^{-1}AP = D \end{array}\right.$$

__Define:__ $\underline{u} = P^{-1} \cdot \underline{y} \iff \underline{y} = P \cdot \underline{u}$

$P^{-1}AP = D$

$AP = PD$

$$\underline{y}' = A \cdot \underline{y}$$
$$= A \cdot P\underline{u}$$
$$\underline{y}' = PD\underline{u}$$
$$P^{-1} \cdot \underline{y}' = P^{-1}PD\underline{u}$$
$$(P^{-1} \cdot \underline{y})' = D \cdot \underline{u}$$
$$\underline{u}' = D\underline{u}$$

$$\begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\boxed{\begin{array}{l} u_1 = C_1 \cdot e^{\lambda_1 t} \\ u_2 = C_2 \cdot e^{\lambda_2 t} \\ \vdots \\ u_n = C_n \cdot e^{\lambda_n t} \end{array}} \Leftarrow \left\{\begin{array}{l} u_1' = \lambda_1 \cdot u_1 \\ u_2' = \lambda_2 \cdot u_2 \\ \vdots \\ u_n' = \lambda_n \cdot u_n \end{array}\right.$$

each diff.eqn. can be solved as linear or separable

$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ in Ex.

$$P^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$
$$= \frac{1}{2} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$u_1 = \frac{1}{2}(-y_1 + y_2)$$
$$u_2 = \frac{1}{2}(y_1 + y_2)$$

$$\underline{y} = P \cdot \underline{u} = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= \underline{v_1} \cdot c_1 e^{\lambda_1 t} + \underline{v_2} \cdot c_2 e^{\lambda_2 t} + \dots$$

$$\underline{y} = \underline{\underline{c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + \dots + c_n \underline{v_n} e^{\lambda_n t}}}$$

Connection between second order diff. eqn's and systems:

Ex: $y'' - 3y' + 2y = 0$

$$\left.\begin{matrix} y_1(t) = y \\ y_2(t) = y' \end{matrix}\right\} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= y'' = 3y' - 2y \\ &= 3y_2 - 2y_1 \end{aligned}$$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -2y_1 + 3y_2 \end{aligned} \qquad \underline{y}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \underline{y}$$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$

$r^2 - 3r + 2$

$\lambda_1 = 1, \ \lambda_2 = 2$

Eigenvectors:

$\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \Rightarrow \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \Rightarrow \underline{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

General case:

$y'' + ay' + by = 0$

$$\left.\begin{matrix} y_1 = y \\ y_2 = y' \end{matrix}\right\} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ay' - by \\ &= -by_1 - ay_2 \end{aligned}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = 0$

$\lambda^2 + a\lambda + b = 0$

$r^2 + ar + b = 0$

$\Downarrow$ two sol. $r_1 \neq r_2 \Rightarrow$ diag.

E_{r_1}: $\begin{pmatrix} -r_1 & 1 \\ * & * \end{pmatrix} \to \underline{v_1} = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$

E_{r_2}: $\begin{pmatrix} -r_2 & 1 \\ * & * \end{pmatrix} \to \underline{v_2} = \begin{pmatrix} 1 \\ r_2 \end{pmatrix}$

$$\underline{y} = \begin{pmatrix} c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \end{pmatrix}$$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \cdot \underline{v_2} e^{\lambda_2 t}$$

$$= c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} = \underline{\underline{\begin{pmatrix} c_1 e^t + c_2 e^{2t} \\ c_1 e^t + 2c_2 e^{2t} \end{pmatrix}}} \to y = \underline{c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

Eq. states and stability for systems

$y' = A \cdot y$: Eq. states (= constant solutions)

$A \cdot y = \underline{0}$
$\Downarrow$
$y_e = \underline{0}$ is an eq. state

Stability:

If $\lambda_1, \lambda_2, \ldots, \lambda_n < 0$, then $y_e = \underline{0}$ is globally asymptotically stable.

$$\underline{y} = C_1 \cdot \underline{v}_1 e^{\lambda_1 t} + \ldots + C_n \cdot \underline{v}_n e^{\lambda_n t}$$

② Difference equations

Ex: $y_{t+1} = y_t + 3$, $y_0 = 1$

we want a closed formula

$y_t = \ldots$ expression in t

$= \underline{\underline{1 + 3t}}$

Recurrence:

$y_0 = 1$

$y_1 = y_0 + 3$

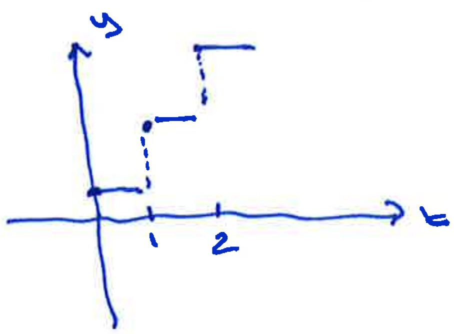
$= 1 + 3 = \underline{4}$

$y_2 = y_1 + 3$

$= \underline{7}$

$\vdots$

Think: $y_t = y(t)$ with discrete time



$y_{t+1} = y_t + 3$

$\underbrace{y_{t+1} - y_t}_{\text{change}} = 3$

(a) First order linear case:

$y_{t+1} + a y_t = f_t$ $\begin{cases} a: \text{const.} \\ f_t: \text{expr. in } t \end{cases}$

Method: Superposition

$y_t = y_t^h + y_t^p =$

Char. eqn: $r + a = 0$, $r = \underline{-a}$ — r^t not e^{rt}

y_t^h: $y_{t+1} + a y_t = 0$

$y_{t+1} = -a \cdot y_t \implies y_t^h = \underline{\underline{C \cdot (-a)^t}}$

y_t^p: $y_{t+1} + a y_t = f_t \leftarrow$ guess y_t

Ex: $y_{t+1} + y_t = 4t$

$y_t = y_t^h + y_t^p = \underline{\underline{C \cdot (-1)^t + 2t - 1}}$

y_t^h: $y_{t+1} + y_t = 0 \implies y_{t+1} = -y_t$

$r + 1 = 0$

$r = -1 \implies y_t^h = \underline{C \cdot (-1)^t}$

y_t^p: $y_{t+1} + y_t = 4t$

$f_t = 4t$

$f_{t+1} = 4(t+1) = 4t + 4$

$(A(t+1) + B) + (At + B) = 4t$

$(2A)t + (A + 2B) = 4t + 0$

$y_t = At + B$

$y_{t+1} = A(t+1) + B$

$2A = 4$ $\quad \underline{A = 2}$

$A + 2B = 0$ $\quad \underline{B = -1}$

$y_t^p = \underline{2t - 1}$

(b) Second order linear case:

Similar method (but char. eqn. is quadratic)

Ex: $y_{t+2} - y_{t+1} - y_t = 0$ ← $y_{t+2} = y_{t+1} + y_t$ Fibonacci seq.

Homogeneous $\Rightarrow y_t = y_t^h$

y_t^h: $r^2 - r - 1 = 0$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

$$\Rightarrow y_t^h = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

Solution: $y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^t$