Plan:

Review Lecture 9

1. First order differential equations
2. Separable diff. equations
3. First order linear diff. equations and the superposition principle
4. Exact differential equations

Will continue with superposition principle and cover exact diff. eqn. next time.

Reading:

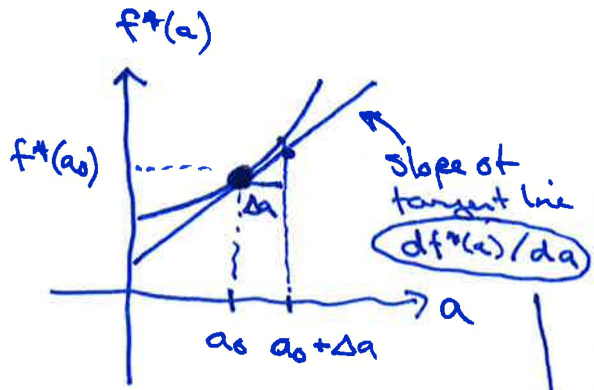
[DE] Differential equations 1.1 – 1.7

Note: Plenary Session 3 Mond. 17-20 in A1-040

Review:

Envelope theorems

change in max/min-value when the parameters of the problem changes



Unconstrained case:

max/min $f(x_1, \ldots, x_n; a)$

$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}\left(x_1^*(a), x_2^*(a), \ldots, x_n^*(a)\right) \longleftarrow$$

where $\underline{x}^*(a) = (x_1^*(a), \ldots, x_n^*(a))$ is the stationary pt. that maximizes $f(\underline{x}; a)$

Estimate:

$$f^*(a_0+\Delta a) \simeq f^*(a_0) + \Delta a \cdot \frac{df^*(a)}{da}$$

we use this at $a = a_0$ to find the slope at the tangent line at $a = a_0$.

Constrained case: Lagrange / Kuhn-Tucker

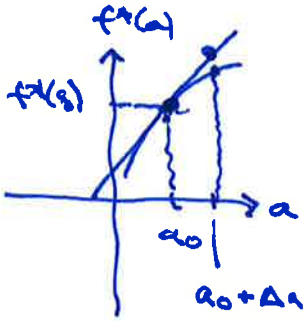
$$\max/\min f(x_1,\dots,x_n;a) \quad \text{when} \begin{cases} g_1(\underline{x};a)=0 \\ \vdots \\ g_m(\underline{x};a)=0 \end{cases} \quad \begin{pmatrix} \le \\ \vdots \\ \le \end{pmatrix}$$

$$L = f(\underline{x};a) - \lambda_1 \cdot g_1(\underline{x};a) - \dots - \lambda_m g_m(\underline{x};a)$$

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\left(\underline{x}^*(a);\ \underline{\lambda}^*(a)\right)$$

where $(\underline{x}^*(a);\underline{\lambda}^*(a))$ is the ordinary candidate point (solving FOC+C / FOC+C+CSC) that is the maximum / minimum pt.

we use this for $a=a_0$



Estimate:

$$f^*(a_0+\Delta a) \simeq f^*(a_0) + \Delta a \cdot \frac{df^*(a)}{da}$$

Kelly Criterion: $(0 \le p \le 1,\ a>0,\ b>0)$

bet with two outcomes, that can be repeated and are independent

$$\text{bet } B \begin{cases} \text{win: } p(W) = p \\ \text{lose: } p(L) = q = 1-p \end{cases}$$

new capital:

$$X_{i+1} = X_i + a \cdot B$$

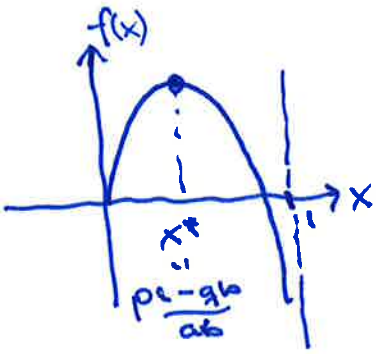
$$X_{i+1} = X_i - b \cdot B$$

Bet: $B_i = x \cdot X_i$

$$0 \le x \le 1$$

maximize {expected exponential growth of capital per bet}

$$\max f(x) = p \cdot \ln(1+ax) + q \cdot \ln(1-bx)$$



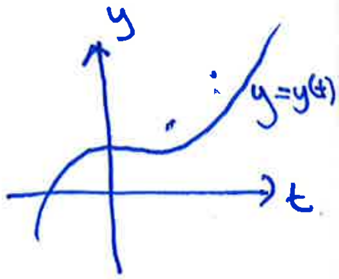
Solution: $x^*(p,a,b) = \dfrac{pa-qb}{ab}$

when $ap - bq > 0$ ← expected return in one bet is positive

1 First order differential equations

Defn:

A <u>differential equation</u> in the unknown function $y=y(t)$ is an equation involving the derivative $y'=y'(t)$, and/or higher order derivatives such as $y''(t), y'''(t), \ldots$



Ex: 1. $y' = y + t \iff y'(t) = y(t) + t$

2. $y' \cdot y = t + 7$

first order differential eqn.

3. $y'' - 7y' + 10y = 4$

second order diff. eqn.

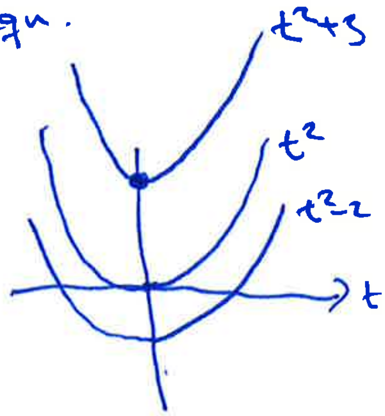
The <u>order</u> of a differential eqn. is the highest order derivative in the diff. eqn.

Ex:

$$y' = 2t$$

$$y = \int 2t\,dt = t^2 + C$$

$\underline{\underline{y = t^2 + C}}$ general solution of the diff. eqn.



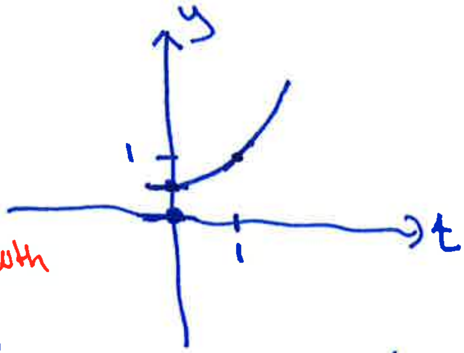
Ex: $\underbrace{y' = 2t}_{\text{diff. eqn.}}$, $\underbrace{y(0) = 3}_{\text{initial cond.}}$ initial value pb.

$\underline{\underline{y = t^2 + 3}}$ particular solution

$y = t^2 + C \qquad y(0) = 3: \quad 3 = 0^2 + C \Rightarrow C = 3$

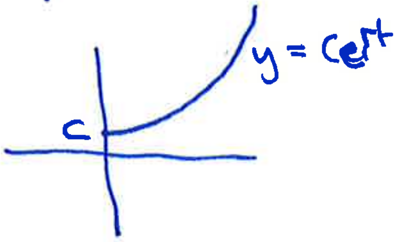
Motivation: $y' = F(t, y)$ first order differential eqn. in std. form.

Ex: $y \cdot y' = t \Rightarrow y' = \frac{t}{y}$



We think of t as time and $y'(t)$ as the rate of change in $y(t)$ = rate of growth

Ex: $y' = ry$ $\quad y = \underbrace{C \cdot e^{rt}}_{\text{general solution}}$



In general: The general solution of a first order differential equation will depend on one undetermined coefficient C.

$$\boxed{\text{Diff. eqn} + \text{initial condition} \Rightarrow \text{unique solution.}}$$

② Separable differential equations

Defn: A first order diff. eqn. is separable if it can be written

$$y' = f(y) \cdot g(t)$$

Ex: $y \cdot y' = t$

$$y' = \frac{t}{y} = \frac{1}{y} \cdot t \quad \leftarrow \text{separable}$$

$$y' - y = t^2$$
$$y' = y + t^2 \quad \leftarrow \text{not separable}$$

Solution method for separable diff. eqn:

Ex: $y' = \frac{1}{y} \cdot t \quad | \cdot y$

$$yy' = t$$
$$\int yy'\,dt = \int t\,dt$$
$$\int y \cdot dy = \frac{1}{2}t^2 + C \quad \text{implicit}$$
$$\frac{1}{2}y^2 + C_1 = \frac{1}{2}t^2 + C_2$$
$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + C_2 - C_1 \quad | \cdot 2$$
$$y^2 = t^2 + 2(C_2 - C_1) = t^2 + K$$
$$y = \pm\sqrt{t^2 + 2(C_2 - C_1)} = \pm\sqrt{t^2 + K}$$

explicit solution

$\boxed{dy = y'\,dt}$

integration by substitution $\boxed{dy = y' \cdot dt}$

General case: $y' = f(y) \cdot g(t) \quad | : f(y)$

$$\frac{1}{f(y)} \cdot y' = g(t)$$
$$\int \frac{1}{f(y)} y'\,dt = \int g(t)\,dt$$
$$\int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$
$$\boxed{\int \frac{1}{f(y)}\,dy = \int g(t)\,dt}$$

implicit general solution

$$y' = \frac{dy}{dt}$$
$$dy = y' \cdot dt$$

Ex: $y' = ry$ for a given constant r
(exponential growth)

$\frac{1}{y} \cdot y' = r$ ← separated diff. eqn

$\int \frac{1}{y} \underbrace{y' \, dt}_{dy} = \int r \, dt$

$\int \frac{1}{y} \, dy = \int r \, dt$

$\ln |y| = rt + C \qquad | \, e^{*}$

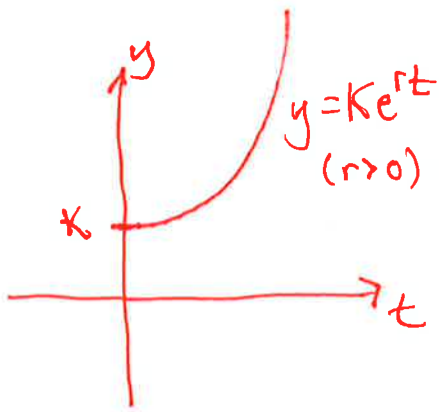
$e^{\ln|y|} = e^{rt+C}$

$|y| = e^{rt} \cdot e^{C}$

$y = \pm e^{rt} \cdot e^{C} = (\pm e^{C}) e^{rt} = K e^{rt}$

$\underline{\underline{y = K \cdot e^{rt}}}$ general solution
exponential growth

$y(0) = K \cdot e^{0r} = K \cdot 1 = K$



(3) Linear differential equations of order one

Defn: A first order differential equation is linear if it can be written

$$\boxed{y' + a(t)\cdot y = b(t)} \iff y' = \underbrace{b(t) - a(t)\cdot y}_{\text{linear in } y}$$

Ex: $yy' = t \Rightarrow y' = \frac{t}{y}$ not linear

$$\boxed{y' + y = t^2} \iff y' = t^2 - y$$ linear

Solution method: Linear differential equations of order one

$$\boxed{y' + a(t)\,y = b(t)}$$

(A) **Integrating factor:** can be used for all first order linear diff eqn.

Ex: $y' + y = t^2 \mid \cdot u$ $\quad a(t) = 1,\ b(t) = t^2$

$(y' + y)\cdot u = t^2 \cdot u$ $\quad u =$ integrating factor

$(uy)' = t^2 \cdot u$ $\quad$ **LHS:** $uy' + uy = (uy)'$

$uy = \int t^2 \cdot u(t)\, dt$ $\quad (uy)' = uy' + u'y$

$y = \frac{1}{u}\cdot \int t^2 \cdot u\, dt$ $\quad$ **Need:** $\underline{u' = u}$

want this!

Can choose: $u = e^t$

see next page

Need to find integrating factor $\underline{u}$ such that $u' = u$.

$$u' = u = u \cdot 1$$

$$\frac{1}{u} u' = 1 \quad \longleftarrow \quad \int \frac{1}{u} \underline{u' dt} = \int \frac{1}{u} du$$

$$\int \frac{1}{u} du = \int 1 \, dt$$

$$\ln |u| = t + C \quad | e^{\cdot}$$

$$|u| = e^{t+C}$$

$$u = \pm e^{t+C} = \pm e^t \cdot e^C$$

$$u = (\pm e^C) \cdot e^t = K e^t$$

$$\underline{u = K e^t} \quad \xleftarrow{K=1}$$

Integrating factor: $\underline{\underline{u = e^t}}$

$\underline{u = e^t}$:

$$y' + y = t^2 \quad | \cdot u = e^t$$

$$(y' + y) e^t = t^2 e^t$$

$$(y \cdot e^t)' = t^2 e^t$$

$$y e^t = \int t^2 e^t \, dt$$

$$y = \frac{1}{e^t} \underbrace{\int t^2 e^t \, dt} = \underline{\underline{t^2 - 2t + 2 + \frac{C}{e^t}}}$$

general solution of $y' + y = t^2$

$$\int \underset{v}{t^2} \underset{u'}{e^t} \, dt = t^2 e^t - \int e^t \cdot 2t \, dt = t^2 e^t - 2 \int \underset{v}{t} \underset{u'}{e^t} \, dt$$

$$= t^2 e^t - 2\left(e^t \cdot t - \int e^t \cdot 1 \, dt\right)$$

$$= \underline{\underline{t^2 e^t - 2t e^t + 2 e^t + C}}$$

$$\boxed{\int u' v \, dt = uv - \int u v' \, dt \quad \text{int. by parts}}$$

In general:

$y' + a(t)y = b(t) \quad | \cdot u$

$(y \cdot u)' = b(t) \cdot u$

$y \cdot u = \int b(t) u \, dt$

$$\boxed{y = \frac{1}{u} \int b(t) \cdot u \, dt, \quad u = e^{\int a(t) dt}}$$

general formula: integrating factor method for linear first order diff. eqn.

Integrating factor:

$u = e^{\int a(t) dt}$

$a(t) = 1$: $e^{\int 1 dt} = e^{t+C} = e^t$

(with $C = 0$)

(B) Superposition principle:

$y' + a \cdot y = b(t)$

Works if $a(t) = a$ is a constant

Homogeneous case: $b(t) = 0$

$y' + ay = 0 \longrightarrow$ { separation, integrating factor

can be solved using these methods

Characteristic equation: $r + a \cdot 1 = 0$

$y' \rightsquigarrow r$
$y \rightsquigarrow 1$

$r = -a \implies y_h = C \cdot e^{rt} = C \cdot e^{-at}$

quick method when a = constant

General solution: $\boxed{y = y_h + y_p}$

y_h: general solution of homogeneous eqn: $y_h = Ce^{-at}$

y_p: a particular solution y_p of the full diff-eqn.

$y' + ay = b(t)$

Will explain this in detail and do examples next time.

Ex: $y' - 2y = 4$ linear first order diff-eqn.

Alt A Integrating factor: $u = e^{\int a(t)\,dt}$:

$y' - 2y = 4$ $a(t) = -2$, $b(t) = 4$

$\int a(t)\,dt = \int -2\,dt = -2t + C \to -2t$

$(y \cdot e^{-2t})' = 4e^{-2t} \leftarrow u = e^{-2t}$ integrating factor

$$ye^{-2t} = \int 4e^{-2t}\,dt = \int 4e^{v} \cdot \frac{dv}{-2} = \frac{4}{(-2)} \int e^{v}\,dv$$

$\boxed{\begin{array}{l} v = -2t \\ dv = -2\,dt \end{array}}$

$$= -2e^{v} + C$$
$$= -2e^{-2t} + C$$

$$\frac{ye^{-2t}}{e^{-2t}} = \frac{-2e^{-2t} + C}{e^{-2t}} \Rightarrow y = -2 + \frac{C}{e^{-2t}} = -2 + Ce^{2t}$$

Alt B Superposition: $y = y_h + y_p = \underbrace{Ce^{2t}}_{y_h} + \underbrace{(-2)}_{y_p} = Ce^{2t} - 2$

y_h: $y' - 2y = 0$ homog. eqn. $r - 2 = 0$, $r = 2$ $\Rightarrow y_h = C \cdot e^{2t}$

y_p: $y' - 2y = 4$ try with $y = $ constant:

$0 - 2A = 4$ $\boxed{\begin{array}{l} y = A \\ y' = 0 \end{array}}$

$-2A = 4$

$A = -2$

$y_p = -2$