Plan:

(1) Introduction to the course
(2) Linear systems and Gaussian elimination
(3) Rank of a matrix

Problems:
See web page.
Exercise session
Mon / Thu
next week.

(1) Introduction: GRA 6035 Mathematics

Topics:
- linear algebra and matrix methods
- optimization in several real variables
- differential and difference equations

Exams:
- midterm exam (1h. multiple choice) - Oct. (20%)
- final exam (3h. written) - End of Nov. (80%)

Material:
- web page
- prep. course FORK 1003
- text book [ME] Simon, Blume: Mathematics for Econ.
- workbook [WB] pdf
- notes on differential eqn. [DE] pdf

Teaching:
- lectures (Fri 08-11) - (weekly)
- plenary sessions (Mon 17-20) - (4 times)
- exercise sessions (Mon/Tue 17-19) - (weekly) most of the time
- office hours (Thu 12-14) - open door policy

How to work with mathematics:
- learn the theory, but more importantly
- work with problems and try to learn from the problems

(2) Linear systems and Gaussian elimination

Defn: A <u>linear system</u> is a system of linear equations. An <u>$m \times n$</u> linear system consists of <u>m</u> equations (linear) in the n variables $x_1, x_2, \dots, x_n$.

An equation in $x_1 \dots x_n$ is <u>linear</u> if it can be written

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_n x_n = b$$

where $a_1, a_2, \dots, a_n, b$ are given numbers.

The general linear system can be written:

$$m \text{ eqn's} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n}_{n \text{ var's}} = b_m \end{array} \right\} \text{general } m \times n \text{ linear system}$$

<u>Geometric interpretation:</u>

<u>$n=2$:</u> $ax + by = c$ ← straight line.

$by = c - ax$
$\downarrow b \neq 0$
$y = c/b - (a/b) \cdot x$

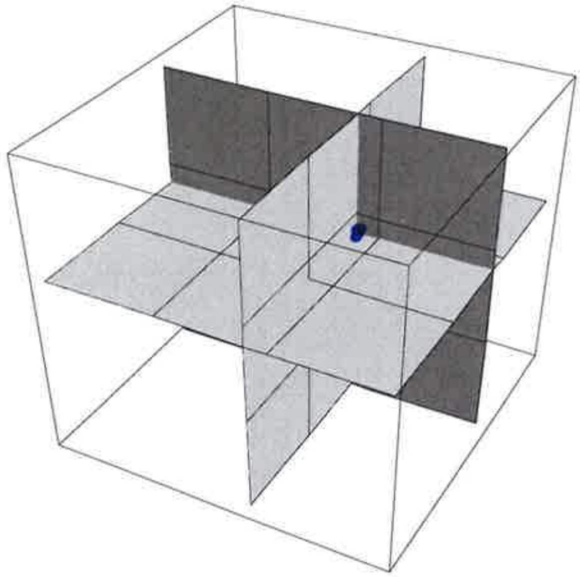
<u>$n=3$:</u> plane in 3D space

(examples next page)

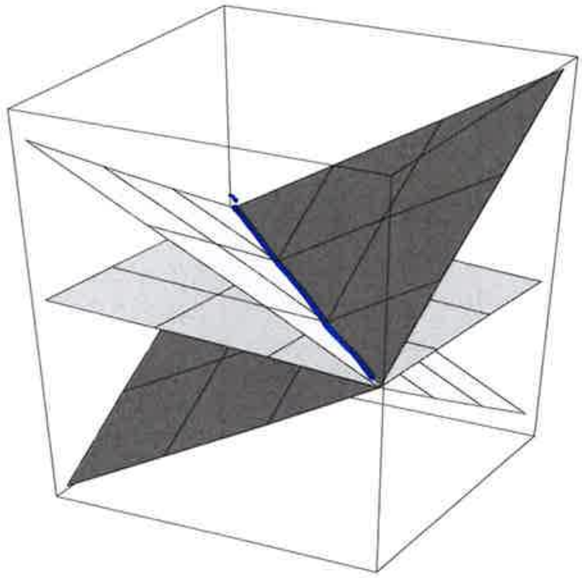
<u>$b=0$:</u> $ax = c$
$(a \neq 0)$ $x = c/a$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

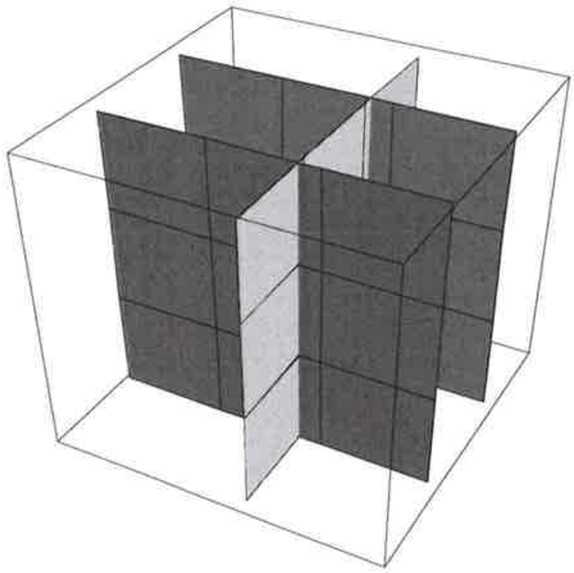
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



Typical example:

3×3 linear system

$$\left.\begin{aligned} x+y-z&=4\\ x-y+z&=2\\ x+2y-z&=6 \end{aligned}\right\}$$

3×3 = 3 eqn's, 3 var's, linear

Theorem A:

Any linear system has either

i) no solutions ← inconsistent lin. system

ii) one unique solution

iii) infinitely many solutions

ii), iii) ← consistent lin. system (at least one solution)

typical example with exactly two solutions – cannot be linear

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n&=b_1\\ \vdots\qquad\qquad & \;\;\vdots\\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n&=b_m \end{aligned}$$

general $m\times n$ lin. system

$$\longrightarrow \quad \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1\\ a_{21} & a_{22} & \dots & a_{2n} & b_2\\ \vdots & & & & \vdots\\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

The first n columns: coefficient matrix of the lin. sys. $(m\times n)$

The whole matrix: augmented matrix of the lin. sys. $m\times(n+1)$-matrix

Gaussian elimination

efficient method for solving any linear system

Ex:
$$x+y-z=4$$
$$x-y+z=2$$
$$x+2y-z=6$$

$$\begin{array}{cccc} x & y & z & \\ \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & -1 & 6 \end{array}\right) & & & \end{array} \begin{matrix} -1 \\ -1 \end{matrix}$$

Want zeros, use pivot in the same col.

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & 1 & 0 & 2 \end{array}\right) \; \tfrac{1}{2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & 2 & -2 \end{array}\right) \; 2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

echelon form

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

pivot = first (left-most) non-zero entry in a row

elementary row operations:

i) interchange two rows

ii) multiply a row with $c \neq 0$

iii) add a multiple of one row to another row

$$R(i) := R(i) + c \cdot R(j)$$

$$\begin{pmatrix} i \\ j \end{pmatrix} \; c$$

echelon form:

i) all zero rows are below all other rows

ii) all entries below a pivot are zero

pivot positions = positions in the matrix where there are pivots in the echelon form.

Gaussian process:

- start with augmented matrix
- use elementary row operations until we get an echelon form.

Theorem B:

Gaussian elim.

a) Any linear system can be transformed into one in echelon form using elementary row operations. The echelon form is not unique, but the pivot positions are unique.

Gauss-Jordan elimination

b) Any linear system can be transformed into one in reduced echelon form (= echelon form s.t. i) all pivots are 1, ii) all entries over a pivot are zero) using elementary row operations. The reduced echelon form is unique.

We find the solutions from the echelon form, using backwards substitution.

Ex:
$$x+y-z=4$$
$$x-y+z=2 \rightsquigarrow \dots \longrightarrow \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right) \end{array}$$
$$x+2y-z=6$$

$$\Downarrow$$

$$x+y-z=4$$
$$y=2$$
$$2z=2$$

Backwards substitution:

$$x+y-z=4 \Rightarrow x=4-y+z=4-2+1=\underline{3}$$
$$y=2 \Rightarrow \underline{y=2}$$
$$2z=2 \Rightarrow \underline{z=1}$$

$$\Downarrow$$

$(x,y,z)=\underline{(3,2,1)}$ one unique solution.

pivot positions = where in the echelon form you find pivots (the positions)

Note: - you can tell how many solutions there are from the pivot positions

- a linear system is inconsistent (no solutions) $\Longleftrightarrow$ there is a pivot position in the last col.

(if and only if)

Ex: $\left(\begin{array}{ccc|c} 1 & +1 & -1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right)$ $\quad 0\cdot x+0\cdot y+0\cdot z=2$ no solution

- if the system is consistent (one or inf. many sol's)

A variable is called basic if there is a pivot in the corresponding col., and free if there is not a pivot.

Ex:

$$\begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 4 & 0 & 7 & 1 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & 7 \end{array}\right)$$

echelon form

basic x basic y basic w

z: free

x, y, w : basic

$$x + 4y + 7w = 1$$
$$y + 3z + 4w = 0$$
$$2w = 7$$

$$x = 65/2 + 12z$$
$$= 1 + 12z + 56 - 49/2$$
$$x = 1 - 4y - 7w$$
$$y = -3z - 4w = -3z - 14$$
$$w = 7/2$$

$$\left.\begin{aligned} x &= 65/2 + 12z \\ y &= -3z - 14 \\ z &= z \quad \text{(free)} \\ w &= 7/2 \end{aligned}\right\}$$

infinitely many solutions

one degree of freedom (z free)

$\shortparallel$

free var's

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12z + 65/2 \\ -3z - 14 \\ z \\ 7/2 \end{pmatrix} = z \cdot \begin{pmatrix} 12 \\ -3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 65/2 \\ -14 \\ 0 \\ 7/2 \end{pmatrix}$$

Consistent systems:

- no free variables : one unique solution
- at least one degree of freedom : infinitely many solutions

3 Rank of a matrix

Definition: The rank of a matrix is the number of pivot positions in the echelon form of the matrix.

Notation: $\text{rk}(A)$ = rank of A.
or $\text{rk}\,A$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ echelon form

$\text{rk}\,A = 3$

Theorem C

Consider an $m \times n$-linear system with coefficient matrix A and augmented matrix $(A|\underline{b})$. Then we have:

i) the system is inconsistent $\Leftrightarrow \text{rk}\,A \neq \text{rk}(A|\underline{b})$

ii) the system is consistent $\Leftrightarrow \text{rk}\,A = \text{rk}(A|\underline{b})$, and in that case

$\#$ degrees of freedom $= n - \text{rk}(A)$

Note: n = # variables
$\text{rk}(A)$ = # basic var's
$n - \text{rk}(A)$ = # free var's

In Particular:
$\text{rk}A = n \Rightarrow$ one solution
$\text{rk}A < n \Rightarrow$ infinitely many solutions

Ex: $x + y = 4$

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{pmatrix} 1 & 1 & 4 \end{pmatrix}$$

echelon form

y free

$x + y = 4 \quad x = 4 - y$

$$\Downarrow$$

$$\boxed{\begin{aligned} x &= 4 - y \\ y &= y \text{ (free)} \end{aligned}}$$

Gaussian

$$\boxed{\begin{aligned} x &= x \text{ (free)} \\ y &= 4 - x \end{aligned}}$$

alternative

degrees of freedom is the same, the free variables can differ.

Note: In Gaussian elimination, we choose to eliminate variables from the left. This is a choice, other alternative exists.

Advantage: standardized, can be carried out with a clear and simple algorithm in all cases

But: other choices are possible

Result (A):

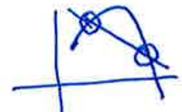
Any $m \times n$ linear system has either

i) one unique solution (consistent)

ii) no solutions (inconsistent)

iii) infinitely many solutions (consistent)

Non-linear case:



two solutions possible

Proof of A:

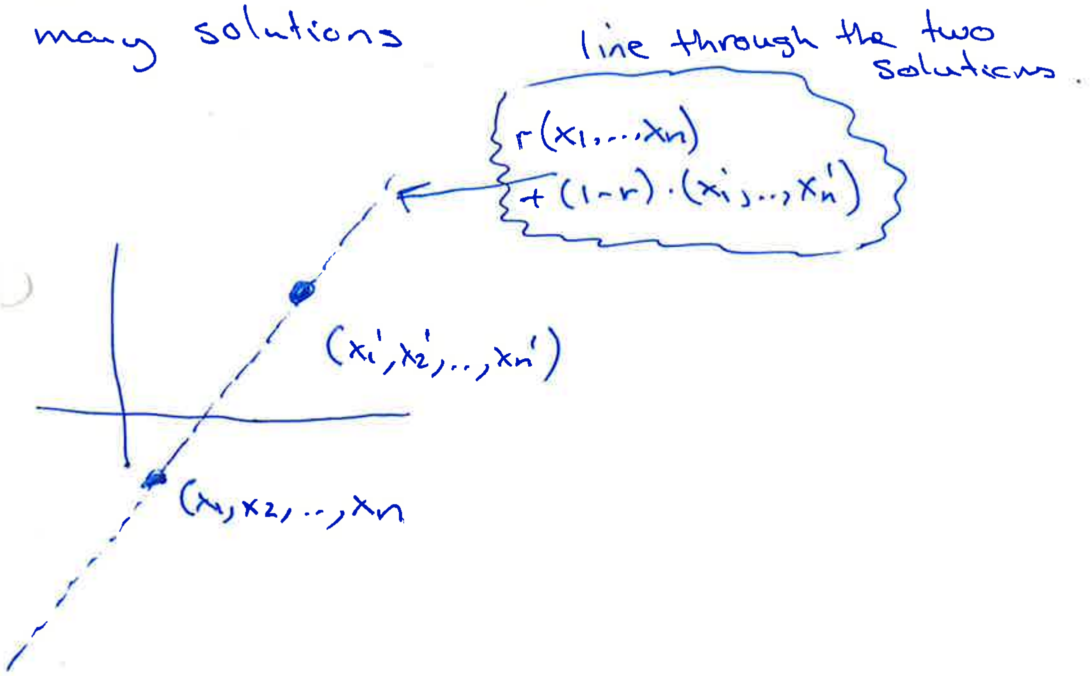
If there are two different solutions

$$(x_1, x_2, \dots, x_n) \neq (x_1', x_2', \dots, x_n')$$

then

$$r(x_1, x_2, \dots, x_n) + (1-r)\cdot(x_1', x_2', \dots, x_n')$$
$$= (r x_1 + (1-r)x_1',\ r x_2 + (1-r) x_2',\ \dots,\ r x_n + (1-r) x_n')$$

is a solution for any number r. There are therefore infinitely many solutions



This can be checked directly. Equation #i is satisfied since

$$a_{i1}\cdot(r x_1 + (1-r)x_1') + a_{i2}(r x_2 + (1-r)x_2') + \dots + a_{in}\cdot(r x_n + (1-r)x_n')$$
$$= r\cdot(a_{i1}x_1 + \dots + a_{in}x_n) + (1-r)\cdot(a_{i1}x_1' + \dots + a_{in}x_n')$$
$$= r\cdot b_i + (1-r) b_i = b_i$$

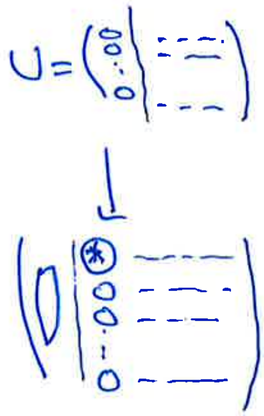
since $(x_1, \dots, x_n)$ and $(x_1', \dots, x_n')$ are solutions

So this equation holds for all i. Hence we have solution for all r. $\square$

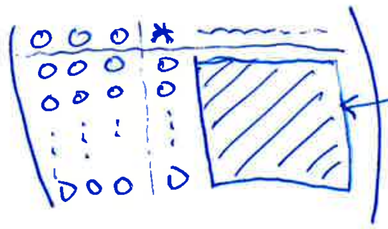
Proof of Result B and more details of Gauss / Gauss-Jordan elimination:

i) Any matrix can be reduced to an echelon form using Elementary row operations.

Start with any matrix U. Move to the right of any columns with only zeros, if any. Look at the first non-zero column, switch two rows in necessary to get a non-zero entry in the top corner. This is a pivot. Use it to get zeros under it.



Now look away from the first row, and look at the remaining part of the matrix.



Repeat the steps above. Since the new matrix is smaller than the original (one row less), we sooner or later get an echelon form this way.

Multiply each row with $\frac{1}{\text{pivot}}$ to set pivot $= 1$. Use each pivot to set zeros over it, starting from the rightmost. You get a reduced echelon form.

ii) If a matrix A can be reduced to reduced echelon forms U, V using elementary row operations, $U = V$.

We have $(A \mid 0) \dashrightarrow (U \mid 0)$ and $(A \mid 0) \dashrightarrow (V \mid 0)$, and elementary row operations do not change solutions of linear systems. So $U \cdot \underline{x} = \underline{0}$ and $V \cdot \underline{x} = \underline{0}$ have the same solutions.

Write $U = (c_1 | c_2 | \dots | c_n)$ and $V = (c_1' | c_2' | \dots | c_n')$ in terms of their columns. We have

$$c_i = x_1 c_1 + x_2 c_2 + \dots + x_{i-1} c_{i-1} \Leftrightarrow c_i \text{ non-pivot column in } U$$

$$\Updownarrow$$

$$c_i' = x_1 c_1' + x_2 c_2' + \dots + x_{i-1} c_{i-1}' \Leftrightarrow c_i' \text{ non-pivot column in } V$$

So U and V have the same pivot columns; They are in the positions, and the pivot columns are

$$\begin{array}{cccc} 1 & 2 & 3 & & r \\ \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, & \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, & \begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix}, & \cdots, & \begin{pmatrix}0\\0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}\end{array}$$

To show that $U=V$, we must show that non-pivot columns are equal. But each non-pivot column satisfy

$$\begin{cases} C_i = x_1 C_1 + \dots + \\ C_i' = x_1 C_1' + \dots + \end{cases}$$ (linear combination of pivot columns to the right)

and the pivot columns are equal, and the coeff's x_i are equal.

Hence $U=V$.

iii) You can always get from an echelon matrix to a reduced echelon matrix, using elementary row operations, without changing the pivot positions.

This follows from the last steps in i).

$\square$

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Proof of Thm C: $m \times n$ linear system $\to$ **n variables**

no solutions $\Leftrightarrow$ pivot position in last column $\Leftrightarrow$ $\underline{rk(A|\underline{b}) = rk(A) + 1}$

$> rk(A)$

Consistent: $\Leftrightarrow$ no pivot position in last column $\Leftrightarrow$ $rk(A|b) = rk(A)$.

If it is consistent: n variables

$rk(A)$ pivots $= rk(A)$ basic variables

$\Downarrow$

$n - rk(A)$ free variables