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Problem Session 4

GRA 6035

MATHEMATICS

EIVIND ERIKSEN , NOV 15 2016

Problems

10.8 ii), 10.12, 10.14, 10.15

11.5, 11.6 ii), 11.7, 11.8 ii), 11.9, 11.11, 11.14

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10.8 ii) $(1+t^3)y' = t^2y$, $(t_0, y_0) = (0,2)$ $\Updownarrow$ $y(0)=2$

$$y' = \frac{t^2y}{1+t^3} = \frac{t^2}{1+t^3}\cdot y$$

Seperable:

$$\frac{1}{y}\cdot y' = \frac{t^2}{1+t^3}$$

$$\int \frac{1}{y}\,dy = \int \frac{t^2}{1+t^3}\,dt = \int \frac{\cancel{t^2}}{u}\cdot\frac{du}{3\cancel{t^2}}$$

$$\boxed{\begin{aligned} u &= 1+t^3 \\ du &= 3t^2\cdot dt \end{aligned}}$$

$$dt = \frac{du}{3t^2}$$

$$\ln|y| = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+t^3| + C$$

$$|y| = e^{\frac{1}{3}\cdot\ln|1+t^3|+C} = \left(e^{\ln|1+t^3|}\right)^{1/3}\cdot e^C$$

$$|y| = |1+t^3|^{1/3}\cdot e^C$$

$$y = K\cdot(1+t^3)^{1/3} = \underline{\underline{K\sqrt[3]{1+t^3}}}$$

$y(0)=2$: $t=0$, $y=2$

$$2 = K\cdot\sqrt[3]{1+0} = K$$

$$\underline{K=2}$$

$$y = \underline{\underline{2\cdot\sqrt[3]{1+t^3}}}$$

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linear:

$$y' = \frac{t^2}{1+t^3} \cdot y$$

$$y' - \frac{t^2}{1+t^3} y = 0$$

$a(t) = -\frac{t^2}{1+t^3}$

$b(t) = 0$

Int. factor:

$\int a(t)\,dt$

$= \int -\frac{t^2}{1+t^3}\,dt$

$= -\frac{1}{3}\ln|1+t^3| + C$

$v = e^{-\frac{1}{3}\ln|1+t^3|}$

$= \left(e^{\ln|1+t^3|}\right)^{-1/3}$

$= |1+t^3|^{-1/3}$

$$(y \cdot v)' = 0$$

$$y \cdot v = C$$

$$y = \frac{C}{v} = \frac{C}{|1+t^3|^{-1/3}}$$

$$y = C \cdot |1+t^3|^{1/3}$$

$y(0) = 2$: $\quad 2 = C \cdot 1^{1/3}$

$C = 2$

$$y = 2 \cdot |1+t^3|^{1/3}$$

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10.12.

i) $ty' + 2y + t = 0 \qquad (t \neq 0)$

$y' = -\frac{2y+t}{t}$ not seperable

$y' = -\frac{2}{t} \cdot y - 1$

$y' + \frac{2}{t} y = -1$ linear $a(t) = 2/t$, $b(t) = -1$

$(y \cdot t^2)' = -t^2$ Int. faktor:

$y \cdot t^2 = \int -t^2 \, dt$ $\int 2/t \, dt = 2\ln t + C$

$yt^2 = -\frac{1}{3}t^3 + C$ $v = e^{2\ln t} = \underline{\underline{t^2}}$

$y = -\frac{1}{3}t + \frac{C}{t^2}$ $(e^{\ln t})^2 = t^2$

ii) $y' - y/t = t \qquad (t > 0)$

$y' - \frac{1}{t} y = t$ $v = e^{\int -1/t \, dt}$

$(y \cdot \frac{1}{t})' = t \cdot \frac{1}{t} = 1$ $= e^{-\ln t} = t^{-1} = \frac{1}{t}$

$(e^{\ln t})^{-1} = t^{-1}$

$y \cdot \frac{1}{t} = \int 1 \, dt = t + C$

$y = \underline{\underline{t^2 + Ct}}$

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<u>10.15.</u> $2t+y-(4y-t)y'=0 \quad , \quad y(0)=0$

$$2t+y=(4y-t)y'$$

$$y'=\frac{2t+y}{4y-t}$$ not sep. not lin.

<u>Exact?</u> $(2t+y)-(4y-t)y'=0$

<u>Look for</u> h s.t. $h'_t=2t+y$, $h'_y=-4y+t$

$h = h(y,t)$

$$\boxed{\begin{aligned} h'_t &= 2t+y \\ h'_y &= -4y+t \end{aligned}}$$

$$h'_t=2t+y \Rightarrow h=\underline{t^2+yt+C(y)}$$

$$h'_y=0+\underline{t}+C'(y)=-4y+\underline{t}$$

$$C'(y)=-4y$$

$$C(y)=\underline{-2y^2}$$

<u>Concl:</u> The diff. eqn. is exact: $h=t^2+yt-2y^2$

<u>General sol:</u> $h=C$

$$t^2+yt-2y^2=C$$

$$\underbrace{-2}_{a}y^2+\underbrace{t}_{b}\cdot y+\underbrace{(t^2-C)}_{c}=0$$

$$y=\frac{-t\pm\sqrt{t^2-4(-2)\cdot(t^2-C)}}{-4}$$

$\underline{y(0)=0}$: $0=0\pm\frac{\sqrt{-8C}}{4}$ $\quad \underline{C=0}$

$$y=\frac{t}{4}\pm\frac{\sqrt{t^2+8(t^2-C)}}{4}$$

$$=\frac{t}{4}\pm\frac{\sqrt{9t^2}}{4}=\frac{t\pm3t}{4}$$

$\underline{\underline{y=t}}$ or $\underline{\underline{y=-\frac{t}{2}}}$

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10.14. $2t + 3y^2 \cdot y' = 0$

Separable:

$3y^2 \cdot y' = -2t$

$y' = -\frac{1}{3y^2} \cdot 2t$

$3y^2 \cdot y' = -2t$

$\int 3y^2 dy = \int -2t\, dt$

$y^3 = -t^2 + C$

$y = \sqrt[3]{C - t^2}$

Exact:

$\underbrace{2t}_{h'_t} + \underbrace{3y^2}_{h'_y} \cdot y' = 0$

$\boxed{\begin{aligned} h'_t &= 2t \\ h'_y &= 3y^2 \end{aligned}}$

$h = t^2 + C(y)$

$(t^2 + C(y))'_y$

$= 0 + C'(y) = 3y^2$

$C'_y = 3y^2$

$C = y^3$

$h = t^2 + y^3 = C$

$y^3 = C - t^2$

$y = \sqrt[3]{C - t^2}$

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11.5. $y'' + ay' + by = 0$ when $a^2/4 - b = 0$

$r^2 + ar + b = 0$ char. eqn.

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$r = -\frac{a}{2}$ double root $\Rightarrow$ $a^2 - 4b = 0$

$a^2/4 - b = 0$

$y = u(t) \cdot e^{-\frac{a}{2}t}$; $u = u(t)$, $r = -a/2$

$$y = u \cdot e^{rt}$$
$$y' = u' \cdot e^{rt} + u \cdot e^{rt} \cdot r = (u' + ur) \cdot e^{rt}$$
$$y'' = (u'' + u'r) e^{rt} + (u' + ur) \cdot e^{rt} \cdot r$$
$$= (u'' + 2u'r + ur^2) \cdot e^{rt}$$

$$y'' + ay' + by = 0$$
$$(u'' + 2u'r + ur^2) e^{rt} + a \cdot (u' + ur) e^{rt} + bu e^{rt} = 0$$
$$e^{rt} [u'' + 2u'r + ur^2 + au' + aur + bu] = 0$$
$$u'' + \underbrace{2u'r + au'}_{(2r+a)u' \,=\, (2 \cdot (-\frac{a}{2}) + a) u' \,=\, 0} + \underbrace{ur^2 + aur + bu}_{(r^2 + ar + b)u \,=\, 0} = 0$$

$u'' = 0$ $\quad u' = C \quad u = Ct + D \Rightarrow ue^{rt} = (Ct + D) e^{rt}$

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11.6 ii) $3x'' - 30x' + 75x = 2t + 1$ ($y = x$)

$3y'' - 30y' + 75y = 2t + 1 \quad :3$

$y'' - 10y' + 25y = \frac{2}{3}t + \frac{1}{3}$ inhomogeneous linear

$$y = y_h + y_p = (C_1 + C_2 t)e^{5t} + \frac{2}{75}t + \frac{9}{375}$$

y_h: $y'' - 10y' + 25y = 0$

$r^2 - 10r + 25 = 0$

$r = 5$ double root

$r_1 = 5, \quad r_2 = 5$

$y_h = C_1 \cdot e^{5t} + C_2 t e^{5t}$

$= (C_1 + C_2 t)e^{5t}$

$3y'' - 30y' + 75y = 0$

$3r^2 - 30r + 75 = 0$

$r = \frac{30 \pm \sqrt{30^2 - 4 \cdot 3 \cdot 75}}{6}$

$= 5 \pm 0$

$r_1 = 5, \quad r_2 = 5$

y_p: $y'' - 10y' + 25y = \frac{2}{3}t + \frac{1}{3}$

Guess: $\cancel{f(t) = \frac{2}{3}t + \frac{1}{3}}$, $\cancel{f'(t) = 2/3}$, $\cancel{f''(t) = 0}$

$y = At + B$ (A, B const.)

$y' = A$

$y'' = 0$

$0 - 10 \cdot A + 25 \cdot (At + B) = \frac{2}{3}t + \frac{1}{3}$

$(25A)t + (25B - 10A) = \frac{2}{3}t + \frac{1}{3}$

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$$25A = 2/3 \qquad A = \underline{\frac{2}{75}}$$

$$25B - 10A = 1/3$$

$$25B - 10 \cdot \frac{2}{75} = \frac{1}{3}$$

$$25B = \frac{1}{3} + \frac{20}{75} = \frac{25}{75} + \frac{20}{75} = \frac{45}{75}$$

$$B = \frac{\cancel{45}\,9}{75 \cdot \cancel{25}\,5} = \underline{\frac{9}{375}}$$

$$y_p = At + B = \underline{\frac{2}{75}t + \frac{9}{375}}$$

<u>11.7 2)</u> $x'' + 2x' + x = t^2, \quad x(0) = 0, \quad x'(0) = 1$

$$X = x_n + x_p = (C_1 + C_2 t)e^{-t} +$$

<u>x_n</u>:

$$x'' + 2x' + x = 0$$
$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r_1 = r_2 = \underline{-1}$$

$$x_n = \underline{(C_1 + C_2 t)e^{-t}}$$

<u>x_p</u>:

$$f(t) = t^2 \qquad x = At^2 + Bt + C$$
$$f'(t) = 2t \qquad x' = 2At + B$$
$$f''(t) = 2 \qquad x'' = 2A$$

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$$x'' + 2x' + x = t^2$$

$$(2A) + 2\cdot(2At + B) + (At^2 + Bt + C) = t^2$$

$$At^2 + (4A + B)t + (2A + 2B + C) = t^2$$

$A = 1$ $\qquad$ $A = 1$ $\qquad$ $x_p = t^2 - 4t + 6$

$4A + B = 0$ $\qquad$ $B = -4$

$2A + 2B + C = 0$ $\qquad$ $C = 6$

$$x = x_h + x_p = (C_1 + C_2 t)e^{-t} + t^2 - 4t + 6$$

$x(0) = 0$ $\qquad$ $0 = (C_1 + C_2 \cdot 0)e^{-0} + 0^2 - 4\cdot 0 + 6$

$x'(0) = 1$ $\qquad$ $0 = C_1 + 6$

$C_1 = -6$

$$x' = C_2 \cdot e^{-t} + (C_1 + C_2 t)\cdot e^{-t}\cdot(-1) + 2t - 4$$

$$x' = e^{-t}(C_2 - C_1 - C_2 t) + 2t - 4$$

$$1 = e^{-0}(C_2 - C_1 - C_2 \cdot 0) + 2\cdot 0 - 4$$

$$1 = C_2 - C_1 - 4 = C_2 + 6 - 4 = C_2 + 2$$

$1 = C_2 + 2$ $\qquad$ $C_2 = -1$

$$x = (-6 - t)e^{-t} + t^2 - 4t + 6$$

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11.9. $x'' + 2ax' - 3a^2 x = 100e^{bt}$

$x = x_h + x_p =$

x_h: $x'' + 2ax' - 3a^2 x = 0$

$r^2 + 2ar - 3a^2 = 0$

$$r = \frac{-2a \pm \sqrt{4a^2 - 4 \cdot 1 \cdot (-3a^2)}}{2}$$

$$= -a \pm \frac{\sqrt{16a^2}}{2} = -a \pm \frac{4a}{2}$$

$= -a \pm 2a$

$r_1 = a \qquad r_2 = -3a$

$a = 0$: $\cancel{x} y_h = (C_1 + C_2 t) \cdot e^{0t} = C_1 + C_2 t$

$a \neq 0$: $\cancel{x} y_h = C_1 e^{at} + C_2 e^{-3at}$

x_p: $x'' + 2ax' - 3a^2 x = 100e^{bt}$

$\cancel{f(t) = 100e^{bt}}$

$\cancel{f'(t) = 100e^{bt} \cdot b} = 100b \cdot e^{bt}$

$\cancel{f''(t) = 100b^2 \cdot e^{bt}}$

$$\left.\begin{aligned} x &= A \cdot e^{bt} \\ x' &= Ab \cdot e^{bt} \\ x'' &= Ab^2 e^{bt} \end{aligned}\right.$$

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$$Ab^2 e^{bt} + 2a \cdot (Ab e^{bt}) - 3a^2 (A e^{bt}) = 100 e^{bt}$$

$$(Ab^2 + 2ab A - 3a^2 A) \cancel{e^{bt}} = 100 \cancel{e^{bt}}$$

$$A \cdot (b^2 + 2ab - 3a^2) = 100$$

$\underline{b^2 + 2ab - 3a^2 \neq 0:}$ $\quad A = \dfrac{100}{b^2 + 2ab - 3a^2}$

$\underline{b^2 + 2ab - 3a^2 = 0:}$ $\quad ?$

---

1) $\underline{a \neq 0, \; b^2 + 2ab - 3a^2 \neq 0:}$

$$x = x_h + x_p = C_1 e^{at} + C_2 e^{-3at} + \frac{100}{b^2 + 2ab - 3a^2}$$

2) $\underline{a = 0, \; b^2 + 2ab - 3a^2 \neq 0:}$ $\quad (a = 0, b \neq 0)$

$$x = x_h + x_p = C_1 + C_2 t + \frac{100}{b^2}$$

3) $\underline{a \neq 0, \quad b^2 + 2ab - 3a^2 = 0}$

$$x = x_h + x_p = C_1 e^{at} + C_2 e^{-3at} + \ldots$$

4) $\underline{a = 0, \quad b^2 + 2ab - 3a^2 = 0}$

$$x = x_h + x_p = C_1 + C_2 t + \ldots$$

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$$x'' + 2ax' - 3a^2x = 100e^{bt}$$

Choose $\boxed{x = A \cdot e^{bt}}$

If $b^2 + 2ab - 3a^2 = 0$ then this is not a solution for any A.

So $x = Ae^{bt}$ does not work in this case.

Try: $x = \underline{At \cdot e^{bt}}$

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11.11. $y'' + y' - 6y = te^t$

$$y = y_h + y_p = \underline{\underline{C_1 e^{2t} + C_2 e^{-3t} + \left(-\frac{1}{4}t - \frac{3}{16}\right)e^t}}$$

y_h: $y'' + y' - 6y = 0$

$r^2 + r - 6 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4\cdot(-6)}}{2} = \frac{-1 \pm 5}{2} = 2, -3$$

$y_h = C_1 e^{2t} + C_2 e^{-3t}$

y_p: $\cancel{f(t) = te^t}$

$\cancel{f'(t) = 1\cdot e^t + t\cdot e^t = (t+1)e^t}$

$\cancel{f''(t) = 1\cdot e^t + (t+1)e^t = (t+2)e^t}$

$y = (At+B)e^t$

$y' = Ae^t + (At+B)e^t = (At + A + B)e^t$

$y'' = Ae^t + (At+A+B)e^t = (At + 2A + B)e^t$

$$(At + 2A + B)\cancel{e^t} + (At + A + B)\cancel{e^t} - 6(At+B)\cancel{e^t} = t\cancel{e^t}$$

$$-4At + (3A + 2B - 6B) = t$$

$(-4A) = 1 \qquad A = -1/4$

$3A - 4B = 0 \qquad -4B = -3A = -3(-1/4) = 3/4$

$$B = -\frac{3}{16}$$

$$y_p = (At+B)e^t = \left(-\frac{1}{4}t - \frac{3}{16}\right)e^t$$

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$$:e^t \,\Big|\quad (At+2A+B)e^t + (At+A+B)e^t - 6(At+B) = te^t$$

$$At+2A+B + At+A+B - 6At - 6B = t$$

Common mistake: Guess $y = Ate^t$ ie $\boxed{B=0}$

$$At+2A + \cancel{2}At+A - 6At = t$$

$$A\cdot(-4t+3) = t$$

$$A = \frac{t}{-4t+3} \longleftarrow$$ doesnot work Since A must be a constant.

So $B=0$ does not work.