BI NORWEGIAN BUSINESS SCHOOL
Nydalsveien 37, 0484 Oslo
Mailing address NO-0442 Oslo
Telephone [+47] 46 41 00 00
Telefax [+47] 21 04 80 00
www.bi.edu
study@bi.no
Org. no. 971 22 8865

PROBLEM SESSION 3

GRA 6035

MATHEMATICS

EIVIND ERIKSEN

NOV 08, 2016

Problems from Lecture 9 and Lecture 7-8:

8.10 / 7.11
7.7 / 8.6 / 9.4
9.5
9.6 (i) / 9.7 (ii)
9.9
9.14

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7.11. $\min \underbrace{x^2+y^2+z^2}_{f(x,y,z)}$ when $2x^2+6y^2+3z^2 \geq 36$

$= \max \underbrace{-(x^2+y^2+z^2)}_{-f(x,y,z)}$ when $-(2x^2+6y^2+3z^2) \leq -36$

$$\mathcal{L} = -x^2-y^2-z^2+\lambda\cdot(2x^2+6y^2+3z^2)$$

$\mathcal{L}'_x = -2x+\lambda\cdot 4x = 0$

$\mathcal{L}'_y = -2y+\lambda\cdot 12y = 0$

$\mathcal{L}'_z = -2z+\lambda\cdot 6z = 0$

a) Binding: $2x^2+6y^2+3z^2=36$, $\lambda \geq 0$

b) Non-binding: $2x^2+6y^2+3z^2 > 36$, $\lambda = 0$

$-2x = -2y = -2z = 0$

$x=y=z=0$ $\lambda=0$

$0 > 36$ not possible

⇓

no solutions

$2x\cdot(-1+2\lambda)=0 \Leftrightarrow x=0$ or $\lambda=1/2$

$2y\,(-1+6\lambda)=0 \Leftrightarrow y=0$ or $\lambda=1/6$

$2z\,(-1+3\lambda)=0 \Leftrightarrow z=0$ or $\lambda=1/3$

Solutions:

$x=0, y=0, z=0$ ⇒ $2x^2+6y^2+3z^2=0\neq 36$ → not solution

$x=0,\ y=0,\ \lambda=1/3$ — $3z^2=36 \to z^2=12 \to z=\pm\sqrt{12}$

$x=0,\ \lambda=1/6,\ z=0$ — $6y^2=36 \to y^2=6 \to y=\pm\sqrt{6}$

$\lambda=1/2,\ y=0,\ z=0$ — $2x^2=36$, $x^2=18$, $x=\pm\sqrt{18}$

$x=0,\ \lambda=1/6,\ \lambda=1/3$ not possible

⇓

no more candidates

⇓

$(0,0,\pm\sqrt{12};\ 1/3)$	$f=12$
$(0,\pm\sqrt{6},0;\ 1/6)$	$f=6$
$(\pm\sqrt{18},0,0;\ 1/2)$	$f=18$

Best candidate.

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8.10. SOC for $(0, \pm\sqrt{6}, 0 ; 1/6)$ $f = 6$

$$L(x,y,z, 1/6) = -(x^2+y^2+z^2) + \frac{1}{6}\cdot(2x^2+6y^2+3z^2)$$

$$= -x^2 \cancel{-y^2} - z^2 + \frac{1}{3}x^2 \cancel{+y^2} + \frac{1}{2}z^2$$

$$= -\frac{2}{3}x^2 - \frac{1}{2}z^2 \qquad \text{concave}$$

$$H(L) = \begin{pmatrix} -4/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{array}{ll} \lambda_1 = -4/3 & \le 0 \\ \lambda_2 = 0 & \le 0 \\ \lambda_3 = -1 & \le 0 \end{array}$$

neg. semidefinite

$$\begin{array}{lll} D_1 = -4/3 & \Delta_1 = -4/3,\ 0,\ -1 & \le 0 \\ D_2 = 0 & \Delta_2 = 0,\ 4/3,\ 0 & \ge 0 \quad \text{ok.} \\ D_3 = 0 & \Delta_3 = 0 & \le 0 \end{array}$$

SOC: $(x,y,z) = (0, \pm\sqrt{6}, 0)$ is max for $-f$

$=$ min for f

$f(0, \pm\sqrt{6}, 0) = 6$ is min

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7.7 max $x+4y+z$ when $\begin{cases} x^2+y^2+z^2=216 \\ x+2y+3z=0 \end{cases}$

$$\mathcal{L} = x+4y+z-\lambda_1(x^2+y^2+z^2)-\lambda_2(x+2y+3z)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1-\lambda_1\cdot 2x-\lambda_2 = 0 \\ \mathcal{L}'_y &= 4-\lambda_1\cdot 2y-2\lambda_2 = 0 \\ \mathcal{L}'_z &= 1-\lambda_1\cdot 2z-3\lambda_2 = 0 \\ & x^2+y^2+z^2 = 216 \\ & x+2y+3z = 0 \end{aligned}}$$

$1-\lambda_2 = 2\lambda_1\cdot x$

$x = \frac{1-\lambda_2}{2\lambda_1}$

$4-2\lambda_2 = 2\lambda_1\cdot y$

$y = \frac{4-2\lambda_2}{2\lambda_1}$

$1-3\lambda_2 = 2\lambda_1\cdot z$

$z = \frac{1-3\lambda_2}{2\lambda_1}$

$\lambda_1=0$:

$1-\lambda_2=0$

$\lambda_2=1$

$4-2\lambda_2=0$

$2=0$

imp

$$2\lambda_1\cdot \Big| \; \frac{1-\lambda_2}{2\lambda_1}+2\cdot\frac{4-2\lambda_2}{2\lambda_1}+3\cdot\frac{1-3\lambda_2}{2\lambda_1}=0$$

$$1-\lambda_2+2(4-2\lambda_2)+3(1-3\lambda_2)=0$$

$$12-14\lambda_2=0 \qquad \lambda_2 = 12/14 = 6/7$$

$$\begin{cases} x = \frac{1/7}{2\lambda_1} = \frac{1}{7}\cdot\frac{1}{2\lambda_1} \\ y = \frac{4-12/7}{2\lambda_1} = \frac{16}{7}\cdot\frac{1}{2\lambda_1} \\ z = \frac{1-3\cdot 6/7}{2\lambda_1} = \frac{-11}{7}\cdot\frac{1}{2\lambda_1} \end{cases}$$

$$x^2+y^2+z^2 = 216$$

$$\left(\frac{1}{7}\right)^2\cdot\left(\frac{1}{2\lambda_1}\right)^2+\left(\frac{16}{7}\right)^2\cdot\left(\frac{1}{2\lambda_1}\right)^2+\left(\frac{-11}{7}\right)^2\left(\frac{1}{2\lambda_1}\right)^2=216$$

$$\left(\frac{1}{49}+\frac{256}{49}+\frac{121}{49}\right)\cdot\left(\frac{1}{4\lambda_1^2}\right)=216$$

$$\frac{378}{49}\cdot\frac{1}{4\lambda^2}=216 \rightarrow 4\lambda^2=\frac{378}{49\cdot 216} \Rightarrow \lambda^2=\frac{378}{4\cdot 49\cdot 216} \Rightarrow$$

$$\lambda = \pm\sqrt{\frac{378}{4\cdot 49\cdot 216}} = \pm\frac{1}{4}\cdot\frac{1}{\sqrt{7}} = \pm\frac{\sqrt{7}}{28}$$

$$\frac{1}{2\lambda_1} = \frac{1}{2}\cdot\left(\pm\frac{28}{\sqrt{7}}\right) = \pm\frac{14}{\sqrt{7}} = \pm\frac{2\cdot 7}{\sqrt{7}} = \underline{\pm 2\sqrt{7}}$$

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$$x = \pm \frac{1}{7} \cdot 2\sqrt{7} = \pm \frac{2}{\sqrt{7}}$$

$$y = \pm \frac{16}{7} \, 2\sqrt{7} = \pm \frac{32}{\sqrt{7}}$$

$$z = \pm \left(-\frac{11}{7}\right) \cdot 2\sqrt{7} = \mp \frac{22}{\sqrt{7}}$$

$$\left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, -\frac{22}{\sqrt{7}}; \frac{\sqrt{7}}{28}, \frac{6}{7} \right)$$

$$\left(-\frac{2}{\sqrt{7}}, -\frac{32}{\sqrt{7}}, \frac{22}{\sqrt{7}}; -\frac{\sqrt{7}}{28}, \frac{6}{7} \right)$$

8.6. $f(x,y,z) = x + 4y + z$

$$f\left(\frac{2}{\sqrt{7}}, \frac{32}{\sqrt{7}}, -\frac{22}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}} \cdot (130 - 22) = \frac{108}{\sqrt{7}} \longleftarrow \text{Best cand. for max}$$

$$f\left(-\frac{2}{\sqrt{7}}, -\frac{32}{\sqrt{7}}, \frac{22}{\sqrt{7}}\right) = \frac{1}{\sqrt{7}} \cdot (22 - 130) = -\frac{108}{\sqrt{7}}$$

SOC: $L = x + 4y + z - \frac{\sqrt{7}}{28} \cdot (x^2 + y^2 + z^2) - \frac{6}{7}(x + 2y + 3z)$

$$H(L) = \begin{pmatrix} -\frac{\sqrt{7} \cdot 2}{28} & 0 & 0 \\ 0 & -\frac{\sqrt{7} \cdot 2}{28} & 0 \\ 0 & 0 & -\frac{\sqrt{7} \cdot 2}{28} \end{pmatrix} \Rightarrow L \text{ concave}$$

neg. defn

$\left(2/\sqrt{7}, 32/\sqrt{7}, -22/\sqrt{7} \right)$ is max.

$f = \frac{108}{\sqrt{7}}$ max value.

9.4. Estimate max value of the problem:

max $x + 4y + z$ when

$$\begin{cases} x^2 + y^2 + z^2 = 215 \\ x + 2y + 3z = 0.1 \end{cases}$$

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$$\max \quad x+4y+z \quad \text{whn} \begin{cases} x^2+y^2+z^2 = a & \to x^2+y^2+z^2-a=0 \\ x+2y+3z = b & x+2y+3z-b=0 \end{cases}$$

We know: $a=216 \quad b=0 \qquad f^*(216,0) = \frac{108}{\sqrt{7}}$

$$x^*(216,0) = \frac{2}{\sqrt{7}} \qquad \lambda_1^*(216,0) = +\frac{\sqrt{7}}{28}$$

$$y^*(216,0) = \frac{32}{\sqrt{7}} \qquad \lambda_2^*(216,0) = 6/7$$

$$z^*(216,0) = -\frac{22}{\sqrt{7}}$$

$$f^*(215,0.1) \simeq f^*(216,0) + \frac{df^*(a,b)}{da}\cdot \Delta a + \frac{df^*(a,b)}{db}\cdot \Delta b$$

$$= \frac{108}{\sqrt{7}} + \frac{\sqrt{7}}{28}\cdot(-1) + \frac{6}{7}\cdot(0.1)$$

Envelope thm.

$$\frac{df^*(a,b)}{da} = \frac{\partial L}{\partial a}\left(x^*(a,b), y^*(a,b), z^*(a,b); \lambda_1^*(a,b), \lambda_2^*(a,b)\right)$$

$$\frac{\partial L}{\partial a} = \lambda_1 \rightsquigarrow \lambda_1^*(a,b) = \frac{\sqrt{7}}{28}$$

$$\mathcal{L} = x+4y+z - \lambda_1\cdot(x^2+y^2+z^2-a) - \lambda_2(x+2y+3z-b)$$

$$= x+4y+z - \lambda_1(x^2+y^2+z^2) + \lambda_1 a - \lambda_2(x+2y+3z) + \lambda_2 b$$

$$\frac{\partial L}{\partial b} = \lambda_2 \rightsquigarrow \lambda_2^*(a,b) = 6/7$$

$$\frac{df^*(a,b)}{db} = \frac{\partial L}{\partial b}\left(x^*(a,b), y^*(a,b), \dots\right)$$

$$= \frac{108}{\sqrt{7}} - \frac{\sqrt{7}}{28} + \frac{6}{7}\cdot 0.1 \simeq \underline{\underline{40.811}}$$

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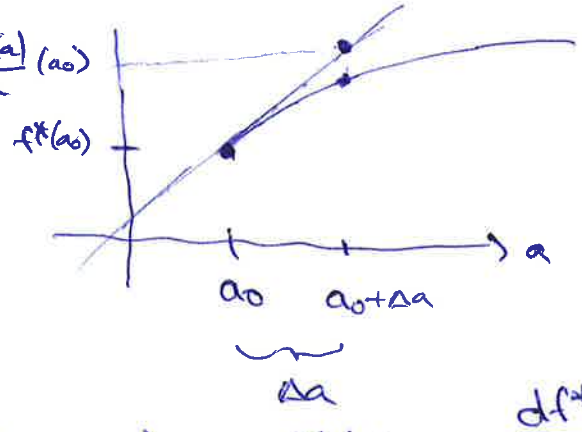
Problems with one parameter: $f^*(a)$ optimal value fn.

$a = a_0$: $f^*(a_0)$ known

$a = a_0 + \Delta a$: $f^*(a) \simeq f^*(a_0) + \frac{df^*(a)}{da}(a_0) \cdot \Delta a$

Estimate: $f^*(a_0) + \Delta a \frac{df^*(a)}{da}(a_0)$



Formula: $f^*(a_0+\Delta a) \simeq f^*(a) + \frac{df^*(a)}{da}(a_0) \cdot \Delta a$

Problems with two parameters:

$$f^*(a_0+\Delta a, b_0+\Delta b) \simeq f^*(a_0, b_0) + \frac{\partial f^*(a,b)}{\partial a}(a_0,b_0) \cdot \Delta a + \frac{\partial f^*(a,b)}{\partial b}(a_0,b_0) \cdot \Delta b$$

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9.5. max $U(x,y) = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y)$

when $2x+3y=m \rightarrow 2x+3y-m=0$

where m is a parameter, $m \geq 4$

i) $L = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) - \lambda\cdot(2x+3y-m)$

$\lambda \neq 0$ $\dots + \lambda m$

$\frac{\partial L}{\partial m} = \lambda$

$$\boxed{\begin{aligned} L'_x &= \frac{1}{2}\,\frac{1}{1+x} - 2\lambda = 0 \\ L'_y &= \frac{1}{4}\cdot\frac{1}{1+y} - 3\lambda = 0 \\ &2x+3y = m \end{aligned}}$$

$\frac{1}{2}\,\frac{1}{1+x} = 2\lambda \qquad \frac{1}{1+x} = 4\lambda \qquad \frac{1}{4\lambda} = 1+x \qquad x = \frac{1}{4\lambda} - 1$

$\frac{1}{4}\,\frac{1}{1+y} = 3\lambda \qquad \frac{1}{1+y} = 12\lambda \qquad \frac{1}{12\lambda} = 1+y \qquad y = \frac{1}{12\lambda} - 1$

$2\left(\frac{1}{4\lambda} - 1\right) + 3\cdot\left(\frac{1}{12\lambda} - 1\right) = m$

$\frac{2\cdot 1}{2\cdot 2\lambda} - 2 + \frac{1}{4\lambda} - 3 = m$

$\frac{2+1}{4\lambda} = m+5 \qquad \frac{3}{4\lambda} = m+5 \qquad \frac{4\lambda}{3} = \frac{1}{m+5} \qquad 1\cdot\frac{3}{4}$

$U = \frac{1}{2}\ln\frac{m+5}{3} + \frac{1}{4}\ln\frac{m+5}{9}$

$\Rightarrow \lambda = \frac{3}{4(m+5)} \rightarrow 4\lambda = \frac{3}{m+5} \qquad 12\lambda = \frac{9}{m+5}$

$x = \frac{1}{4\lambda} - 1 = \frac{m+5}{3} - 1 \qquad y = \frac{1}{12\lambda} - 1 = \frac{m+5}{9} - 1 \qquad \lambda = \frac{3}{4(m+5)}$

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<u>SOC:</u> $x = \frac{m+5}{3} - 1 \quad y = \frac{m+5}{9} - 1 \quad \lambda = \frac{3}{4(m+5)}$

$$\mathcal{L} = \frac{1}{2}\ln(1+x) + \frac{1}{4}\ln(1+y) - \frac{3}{4(m+5)} \cdot (2x+3y-m)$$

$$\mathcal{L}''_{xx} = \left(\frac{1}{2} \cdot \frac{1}{1+x}\right)'_x = \frac{1}{2} \cdot -1 \cdot (1+x)^{-2} = -\frac{1}{2(1+x)^2} < 0$$

$$\mathcal{L}''_{xy} = \left(\frac{1}{2} \cdot \frac{1}{1+x}\right)'_y = 0$$

$$\mathcal{L}''_{yy} = \left(\frac{1}{4} \cdot \frac{1}{1+y}\right)'_y = \frac{1}{4} \cdot (-1) \cdot (1+y)^{-2} = -\frac{1}{4(1+y)^2} < 0$$

$$H(\mathcal{L}) = \begin{pmatrix} -\frac{1}{2(1+x)^2} & 0 \\ 0 & -\frac{1}{4(1+y)^2} \end{pmatrix}$$

neg. detn.

$D_1 = -\frac{1}{2(1+x)^2} < 0$

$D_2 = \frac{1}{8(1+x)^2(1+y)^2} > 0$

$\Rightarrow \mathcal{L}$ <u>concave</u>

$\Downarrow$

$$x^*(m) = \frac{m+5}{3} - 1 \quad y^*(m) = \frac{m+5}{9} - 1 \quad \lambda^*(m) = \frac{3}{4(m+5)}$$

<u>max</u>

$$U^*(m) = \frac{1}{2}\ln\left(\frac{m+5}{3}\right) + \frac{1}{4}\ln\left(\frac{m+5}{9}\right)$$

$$= \frac{1}{2}\left(\ln(m+5) - \ln(3)\right) + \frac{1}{4}\left(\ln(m+5) - \ln(9)\right)$$

$$= \frac{3}{4}\ln(m+5) \cancel{- \frac{1}{2}\ln 3} + \cancel{\frac{1}{4} \cdot 2\ln 3} = \underline{\underline{\frac{3}{4}\ln(m+5)}}$$

2) $\dfrac{dU^*(m)}{dm} = \dfrac{3}{4} \cdot \dfrac{1}{m+5} \cdot 1 = \dfrac{3}{4(m+5)} = \lambda$

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9.6 / 9.7 ii)

$$Q(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$$

when

$$\left.\begin{aligned} x_1+x_2+x_3&=0 \\ x_1+x_2-x_3&=0 \end{aligned}\right\} \text{linear}$$

$$L = \overbrace{x_1^2+x_2^2-x_3^2+4x_1x_3-2x_1x_2}^{\text{quadr.}} - \lambda_1(x_1+x_2+x_3) - \lambda_2(x_1+x_2-x_3)$$

$$L'_{x_1} = 2x_1 + 4x_3 - 2x_2 - \lambda_1 - \lambda_2 = 0$$
$$L'_{x_2} = 2x_2 - 2x_1 - \lambda_1 - \lambda_2 = 0$$
$$L'_{x_3} = -2x_3 + 4x_1 - \lambda_1 + \lambda_2 = 0$$
$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 - x_3 = 0$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & -2 & 4 \\ 1 & 1 & -2 & 2 & 0 \\ 1 & -1 & 4 & 0 & -2 \end{pmatrix}$$

$n=3$ $\quad (-1)^3 = -1$
$m=2$ $\quad (-1)^2 = +1$

$n-m=\underline{1}$

$|B| = 16 > 0$ same sign as $(-1)^m = +1$ $\rightarrow$ local min

$(x_1,x_2,x_3;\lambda_1,\lambda_2) = (0,0,0;0,0)$ is one local min

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Without Border Hessian:

$x_1 + x_2 + x_3 = 0$
$x_1 + x_2 - x_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

y free $\quad x = -y - z = -y$

$-2z = 0 \Rightarrow z = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix}, \quad y \text{ free}$$

$Q = x_1^2 + x_2^2 - x_3^2$
$+ 4x_1x_3 - 2x_1x_2$
$= (-y)^2 + y^2 + 0^2$
$+ 4(-y)\cdot 0 - 2\cdot(-y)y$
$= 2y^2 + 2y^2 = 4y^2 \geq 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y = 0$ min