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PROBLEM SESSION 2

GRA6035

MATHEMATICS

EIVIND ERIKSEN

OCT 25, 2016

Problems from Lecture 7/8

Workbook 6.1 iii) 7.1 v), vi)

7.4/8.3, 7.6/8.5, 7.8/8.7

7.9/8.8, $\boxed{7.11/8.10,\ 8.12}$ ← Later?

8.13

6.1 iii)

$$f = xy^2 + x^3y - xy$$

$$f'_x = y^2 + 3x^2y - y = 0$$

$$f'_y = 2xy + x^3 - x = 0$$

y = 0 or $y + 3x^2 - 1 = 0 \iff y \cdot (y + 3x^2 - 1) = 0$

x = 0 or $2y + x^2 - 1 = 0 \iff x \cdot (2y + x^2 - 1) = 0$

$x = 0$, $y = 0$	$y = 0$, $2y + x^2 - 1 = 0$	$x = 0$, $y + 3x^2 - 1 = 0$	$y + 3x^2 - 1 = 0$, $2y + x^2 - 1 = 0$
$(0,0)$ ✓	$x^2 - 1 = 0$ $x = \pm 1$ $(\pm 1, 0)$ ✓	$y - 1 = 0$ $y = 1$ $(0,1)$ ✓	$y = 1 - 3x^2$ $2(1 - 3x^2) + x^2 - 1 = 0$ $1 - 5x^2 = 0$ $x^2 = 1/5$ $x = \pm\sqrt{1/5}$ $y = 1 - 3 \cdot \frac{1}{5} = \frac{2}{5}$ $\left(\pm\sqrt{\frac{1}{5}}, \frac{2}{5}\right)$ ✓

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7.1. v) $x \geq 0, y \geq 0, \; xy \geq 1$

$xy = 1$
$y = 1/x$

$xy > 1$
$y > 1/x$

closed : all boundary pts are included

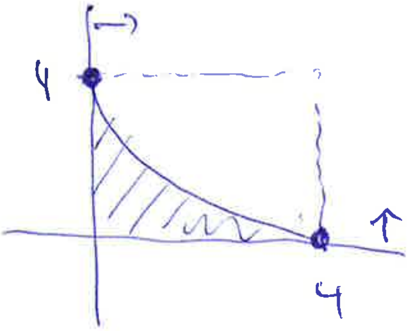
$\geq \quad \leq \quad (=)$

(not open)

not bounded

vi) $\sqrt{x} + \sqrt{y} \leq 2$

closed $\leq$



$\sqrt{x} + \sqrt{y} = 2$

$\sqrt{y} = 2 - \sqrt{x}$

$y = (2 - \sqrt{x})^2$

$= 4 - 2\sqrt{x} + x,$

$x \geq 0$

$y \geq 0$

$x \leq 4 \quad y \leq 4$

$\Downarrow$

bounded

7.4 / 8.3. max $f = xyz$ when $\begin{cases} x^2 + y^2 = 1 \\ x + z = 1 \end{cases}$

$$\mathcal{L} = xyz - \lambda_1 \cdot (x^2 + y^2) - \lambda_2 (x + z)$$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= yz - \lambda_1 \cdot 2x - \lambda_2 = 0 \\ \mathcal{L}'_y &= xz - \lambda_1 \cdot 2y = 0 \\ \mathcal{L}'_z &= xy - \lambda_2 = 0 \end{aligned} \qquad \begin{aligned} x^2 + y^2 &= 1 \\ x + \cancel{z} &= 1 \end{aligned}}$$

$\lambda_2 = xy$ $\quad$ $\lambda_1 = \dfrac{xz}{2y}$ $\qquad$ $\begin{cases} y^2 = 1 - x^2 \\ z = 1 - x \end{cases}$

$$yz - \frac{xz}{\cancel{2}y} \cdot \cancel{2}x - xy = 0 \quad | \cdot y$$

$$y^2 z - x^2 z - xy^2 = 0$$

$$(1 - x^2) \cdot \underline{(1 - x)} - x^2 \cdot \underline{(1 - x)} - x \cdot \underbrace{(1 - x^2)}_{= \underline{(1-x)}(1+x)} = 0$$

$$(1 - x) \cdot \left[1 - x^2 - x^2 - x \cdot (1 + x) \right] = 0$$

$\underline{x = 1}$ or $-3x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{1 - 4(-3) \cdot 1}}{-6}$$

$$\underline{x = \frac{1 + \sqrt{13}}{-6}}, \quad \underline{x = \frac{1 - \sqrt{13}}{-6}}$$

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Candidate pts:

$x=1,\ y=0,\ z=0,\quad \lambda_1 = \cancel{\frac{xz}{2y}}=0 \quad \lambda_2=0 \quad f=0$

$(x,y,z) \approx (0{,}4343,\ \pm 0{,}9008;\ 0{,}5657)$
$(-0{,}7676,\ \pm 0.6409,\ 1{,}7676)$

$y=0$: $x=\pm 1$ $\quad x=1 \Rightarrow z=0 \quad z = 1-(\pm 1)$
~~$x=-1 \Rightarrow z=2$~~

$\lambda_2 = 0 \quad xz=0 \quad -\lambda_1\cdot 2x=0$
$x=1,\ z=0 \quad \lambda_1=0$

Cand. pts with $y=0$:
$x=1,\ y=0,\ z=0;\ \lambda_1=0,\ \lambda_2=0$

8.3. $(x,y,z)\approx(-0{,}7676,\ -0{,}6409,\ 1{,}7676)$ best candidate pt.
$= $ higest value for $f=xyz$

$\lambda_1 = \frac{xz}{2y} \quad \lambda_2 = xy$

SOC: $\mathcal{L} = xyz - \lambda_1\cdot(x^2+y^2) - \lambda_2(x+z)$
doesn't work.

EVT: set of adm pts bounded? Yes $\leftarrow$ $x^2+y^2=1$, $x+z=1$

$-1\le x\le 1$
$-1\le y\le 1$
$0\le z\le 2$, $z = 1-x$

$\Downarrow$ EVT
There is a max.
$\Downarrow$
The best cand. pt. is max if NDCQ holds

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NDCQ: $x^2+y^2=1$, $x+z=1$ $\quad \operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} = 2$

NDCQ fails : $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix} < 2$

$$\Updownarrow$$

$$\begin{vmatrix} 2x & 2y \\ 1 & 0 \end{vmatrix} = 0 \quad \begin{vmatrix} 2x & 0 \\ 1 & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} 2y & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$-2y=0 \quad 2x=0 \quad 2y=0$

$$\Updownarrow$$

$(x,y)=(0,0)$ not adm.

NDCQ holds for all adm pts.

$$\Downarrow$$

$x \cong -0{,}7676 \quad y \cong -0{,}6409 \quad z \cong 1{,}7676$ **max**

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7.6/8.5: max $xz+yz$ when $y^2+z^2=1$, $xz=3$

$= \max 3+yz$ when $y^2+z^2=1$

$\mathcal{L} = 3+yz-\lambda\cdot(y^2+z^2)$

$\mathcal{L}'_y = z-\lambda\cdot 2y = 0$ $\quad y^2+z^2=1$

$\mathcal{L}'_z = y-\lambda\cdot 2z = 0$

$z=\lambda\cdot 2y$

$y = 2\lambda\cdot z = 2\lambda\cdot 2\lambda\cdot y$

$y = 4\lambda^2 y \longrightarrow y=0$ or $\lambda=\pm 1/2$ ($4\lambda^2=1$)

$y=0$: $z=0$, imp. $y^2+z^2=0\neq 1$, no sol's

$\lambda=1/2$: $y=z$, $2z^2=1$, $z^2=1/2$, $z=\pm\sqrt{1/2}$

$\lambda=-1/2$: $z=-y$, $2z^2=1$, $z^2=1/2$, $z=\pm\sqrt{1/2}$

Candidate pts:

$y=z=\pm\sqrt{1/2}$ $\quad \lambda=1/2$ $\quad f=7/2$

$y=-z=\pm\sqrt{1/2}$ $\quad \lambda=-1/2$ $\quad f=5/2$

Best cand: $(\sqrt{1/2},\sqrt{1/2})$, $(-\sqrt{1/2},-\sqrt{1/2})$ $\quad \lambda=1/2$ max

8.5: $\mathcal{L} = 3+yz-\frac{1}{2}(y^2+z^2)$ concave? Yes

$H(\mathcal{L}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad D_1=-1\leq 0$, $D_2=0\geq 0$ $\quad \Delta_1=-1,-1<0$

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7.8/8.7: max $f = xy$ when $x^2+y^2 \le 1$ std. form

$\mathcal{L} = xy - \lambda \cdot (x^2+y^2)$

	FOC	C	CSC
$\mathcal{L}'_x =$	$y - \lambda \cdot 2x = 0$	$x^2+y^2 \le 1$	$\lambda \ge 0$
$\mathcal{L}'_y =$	$x - \lambda \cdot 2y = 0$		$\lambda \cdot (x^2+y^2-1) = 0$

$x^2+y^2=1$	$x^2+y^2<1$
$y = 2\lambda x$	$y = 2\lambda \cdot x$
$x = 2\lambda y$	$x = 2\lambda \cdot y$
$\lambda \ge 0$	$\lambda = 0$
$\Downarrow$	$\Downarrow$
$x = 2\lambda \cdot 2\lambda x$	$x = y = 0,\ \lambda = 0$
$x = 4\lambda^2 \cdot x$	$x^2+y^2 = 0 < 1$ ok.
$x = 0$ or $4\lambda^2 = 1$	$\Downarrow$
$\lambda^2 = 1/4$	$(0,0)$ $\lambda = 0$ $f = \underline{0}$
$\lambda = 1/2 > 0$	

$x = y = \pm\sqrt{1/2}$

$(\sqrt{1/2}, \sqrt{1/2})$ $\lambda = 1/2$ $f = 1/2$

$(-\sqrt{1/2}, -\sqrt{1/2})$ $\lambda = 1/2$ $f = 1/2$

Best cand. pts.

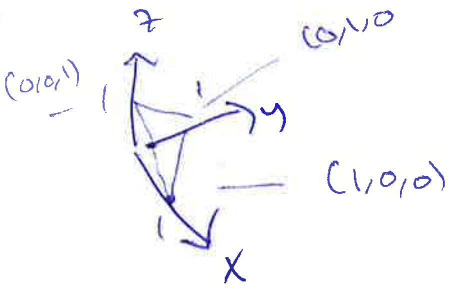
max pts.

8.7: $\mathcal{L} = xy - \frac{1}{2}(x^2+y^2)$

$H(\mathcal{L}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $D_1 = -1,\ \Delta_1 = -1$, $D_2 = 0$

$\mathcal{L}$ concave

7.9/8.8. max xyz when $\begin{cases} x+y+z \le 1 \\ x \ge 0 \iff -x \le 0 \\ y \ge 0 \quad\quad -y \le 0 \\ z \ge 0 \quad\quad -z \le 0 \end{cases}$



Main case $x+y+z=1$, $x,y,z>0$

$$\mathcal{L} = xyz - \lambda_1 \cdot (x+y+z) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

	FOC	C	CSC
$\mathcal{L}'_x =$	$yz - \lambda_1 + \lambda_2 = 0$	$x+y+z \le 1$	$\lambda_1 \ge 0$, $\lambda_1(x+y+z-1)=0$
$\mathcal{L}'_y =$	$xz - \lambda_1 + \lambda_3 = 0$	$x \ge 0$	$\lambda_2 \ge 0$, $\lambda_2 \cdot x = 0$
$\mathcal{L}'_z =$	$xy - \lambda_1 + \lambda_4 = 0$	$y \ge 0$	$\lambda_3 \ge 0$, $\lambda_3 y = 0$
		$z \ge 0$	$\lambda_4 \ge 0$, $\lambda_4 z = 0$

Main case:

$x+y+z=1$ $\quad \lambda_1 \ge 0$

$x>0, y>0, z>0$ $\quad \lambda_2=\lambda_3=\lambda_4=0$

FOC:

$yz - \lambda_1 = 0 \quad \lambda_1 = yz$

$xz - \lambda_1 = 0 \quad \lambda_1 = xz$

$xy - \lambda_1 = 0 \quad \lambda_1 = xy$

$yz = xz$ and $xz = xy$
$\underline{x=y}$ $\quad \underline{y=z}$

$\left.\begin{matrix} x=y=z \\ x+y+z=1 \end{matrix}\right\} \Rightarrow x=y=z=\underline{1/3}$

$\underline{(x,y,z) = (1/3, 1/3, 1/3) \quad \lambda_1 = \frac{1}{9} \quad \lambda_2=\lambda_3=\lambda_4=0}$ $\quad f = \frac{1}{27}$

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$x+y+z=1$
$x,y,z>0$
$\Downarrow$
$(1/3, 1/3, 1/3)$ $f=1/27$

$x+y+z<1$
$x,y,z>0$

$\lambda_2=\lambda_3=\lambda_4=0$
$\lambda_1=0$
$yz=0$ FOC
$xz=0$
$xy=0$
impossible

no cand. pts.

$x=0$ or $y=0$ or $z=0$
$\Downarrow$
$f=xyz=0$ not max

Concl:
$(1/3, 1/3, 1/3)$ best cand. pt.

88. SOC: $L = xyz - \lambda_1\cdot(x+y+z) + \lambda_2 x + \lambda_3 y + \lambda_4 z$

$$H(L) = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

$D_1=0$
$D_2=-z^2$
⋮

not concave

EVT: bounded? Yes.
$\Downarrow$
there is a max

$x+y+z\le 1$
$x\ge 0, y\ge 0, z\ge 0$

$0\le x\le 1$
$0\le y\le 1$
$0\le z\le 1$

NDCQ:
$$J = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

All binding: impossible

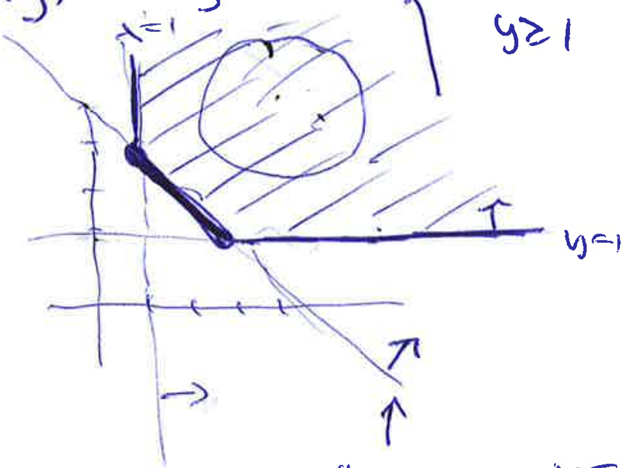
$x+y+z<1$, $x=y=z=0$: $\text{rk}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}=3$ ok.

⋮
ok in all cases.

$\Longrightarrow$ $(1/3, 1/3, 1/3)$ is max

8.13. max $\ln(x^2y) - x - y$ when $\begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$

a) $x+y \geq 4$
$x \geq 1$
$y \geq 1$



$x+y=4$ $\quad x=1 \quad y=1$

$y = 4-x$

b) max $2\ln x + \ln y - x - y$ when $\begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases}$

$$\mathcal{L} = 2\ln x + \ln y - x - y - \lambda_1(-x-y) - \lambda_2(-x) - \lambda_3(-y)$$
$$= 2\ln x + \ln y - x - y + \lambda_1(x+y) + \lambda_2 x + \lambda_3 y$$

	FOC	C	CSC
$\mathcal{L}'_x =$	$\frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0$	$x+y \geq 4$	$\lambda_1 \geq 0,\ \lambda_1(x+y-4) = 0$
$\mathcal{L}'_y =$	$\frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0$	$x \geq 1$	$\lambda_2 \geq 0,\ \lambda_2(x-1) = 0$
		$y \geq 1$	$\lambda_3 \geq 0,\ \lambda_3(y-1) = 0$

Main case:

$x+y=4$ $\quad \lambda_1 \geq 0$ $\quad \frac{2}{x} - 1 + \lambda_1 = 0$ $\quad \lambda_1 = 1 - \frac{2}{x}$

$x>1,\ y>1$ $\quad \lambda_2 = \lambda_3 = 0$ $\quad \frac{1}{y} - 1 + \lambda_1 = 0$ $\quad \lambda_1 = 1 - \frac{1}{y}$

$\Rightarrow \cancel{1} + \frac{2}{x} = \cancel{1} + \frac{1}{y}$ $\quad 2y = x$ $\quad 2y + y = 4$

$3y = 4$

$y = 4/3$ $\quad x = 8/3$

$$\boxed{x = 8/3 \quad y = 4/3 \quad \lambda_1 = \frac{-2}{8/3} + 1 = \frac{-6}{8} + 1 = 1/4 > 0 \quad \lambda_2 = \lambda_3 = 0}$$

Cand. pt. $\quad f =$

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$$\mathcal{L} = 2\ln x + \ln y - x - y + \frac{1}{4}(x+y)$$

$$H(\mathcal{L}) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$$

$D_1 = -2/x^2 < 0$

$D_2 = 2/x^2y^2 > 0$

$\mathcal{L}$ concave

$\Downarrow$

$(x,y) = (8/3, 4/3)$ is max

If I had started with other cases than

$x+y=4$
$x>1, y>1$

I would have to continue with more cases until I found the max.