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PLENARY SESSION 1

GRA6035

MATHEMATICS

EIVIND ERIKSEN

OCT 4, 2016

Plan:

1. Review: a) Determinant, rank and linear independence
   b) Diagonalization / Markov chains
   c) Convex / concave functions and definiteness
2. Problems:
   a) 3.12, Midterm 03/2016 Pb 3.
   b) 4.3b, 4.10, 5.7, 5.9, Midterm 03/2016 Pb. 6, 10/2014 Pb. 5
   c) 6.12, Midterm 10/2014 Pb. 7-8, 03/2016 Pb 7.
3. Exam problem: Midterm 10/2015 Problem 7-8.

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9 Determinant / rank

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n} \rightsquigarrow A = (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})$

Find the rank of A.

(a) If A is square: $|A| \neq 0$: $rk\,A = n \iff$ $\underline{v_1}, \ldots, \underline{v_n}$ are linearly indep.

$|A| = 0$: $rk\,A < n \iff$ $\underline{v_1}, \ldots, \underline{v_n}$ are linearly dep.

(b) If A is not square: Either use row operations ($rk\,A$ = # pivot positions)

or use minors

With parameters: Use determinants

Non-square: Use row-operations

When you find the pivot positions in A:

vectors corresponding to pivot col's: lin indep.

— " — to non-pivot col's: linear combinations of

3.12. A : $m \times n$-matrix A^TA : $n \times n$-matrix

i) Nullspace(A) = Nullspace(A^TA)

$\{$sol'ns of $A\underline{x} = \underline{0}\}$ $\quad$ $\{$sol'ns of $(A^TA)\underline{x} = \underline{0}\}$

Assume:

$A\underline{x} = 0 \Rightarrow A^T \cdot A\underline{x} = A^T \cdot \underline{0} = \underline{0} \Rightarrow A^TA \cdot \underline{x} = \underline{0}$

Assume:

$A^TA\underline{x} = 0 \Rightarrow \underbrace{\underline{x}^T \cdot A^TA\underline{x}}_{(A\underline{x})^T \cdot (A\underline{x})} = \underline{x}^T \cdot \underline{0} = 0$

$(A\underline{x})^T \cdot (A\underline{x}) = (y_1\ y_2 \ldots y_n) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = y_1^2 + y_2^2 + \ldots + y_n^2 = 0$

$\Leftarrow y_1 = 0,\ y_2 = 0,\ \ldots,\ y_n = 0$

$A\underline{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$A\underline{x} = \underline{0}$. ok.

ii) $rk(A) = rk(A^TA)$!

$n - rk(A) = \#$ free var's in $A\underline{x} = \underline{0}$ $\quad$ part i)

$n - rk(A^TA) = \#$ free var's in $A^TA\underline{x} = \underline{0}$

$n - rk(A) = n - rk(A^TA) \Rightarrow \underline{rk(A) = rk(A^TA)}$.

ok.

Midterm 03/2016, Pb 3

$$A = \begin{pmatrix} 1 & 4 & -7 & 3 \\ 2 & -2 & 8 & 0 \\ 2 & 10 & t-16 & 1-t \end{pmatrix} \begin{matrix} -2 \\ \\ -2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 4 & -7 & 3 \\ 0 & -10 & 22 & -6 \\ 0 & 2 & t-2 & -5-t \end{pmatrix} \frac{1}{5}$$

$$\rightarrow \begin{pmatrix} 1 & 4 & -7 & 3 \\ 0 & -10 & 22 & -6 \\ 0 & 0 & * & * \end{pmatrix}$$

$$t-2+\frac{22}{5} \qquad -5-t-\frac{6}{5}$$

$$t+\frac{12}{5} \qquad -t-\frac{31}{5}$$

$t \neq -\frac{12}{5}$: rk $A = 3$

$t = -\frac{12}{5}$: 0 $\qquad \frac{12}{5} - \frac{31}{5} \neq 0$ $\qquad$ rk $A=3$

b) Diagonalization, Markov chains

A $n \times n$-matrix

A diagonalizable

$\Updownarrow$

$P^{-1}AP = D$ for a diagonal D, invertible P

$\Updownarrow$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad P = \left(\underline{v_1} \,\middle|\, \underline{v_2} \,\middle|\, \cdots \,\middle|\, \underline{v_n} \right)$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of A (with correct multiplicity)

$\underline{v_i}$ eigenvector of A with eigenvalue λ_i that are linearly independent

Criterion:

For each $\underline{\lambda_i}$ of mult. m_i, there should be m_i degrees of freedom in $(A - \lambda I)\underline{x} = \underline{0}$.

$\lambda = \lambda_i$

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4.3 b) $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\underbrace{2\cdot(1+1-\lambda)} + \underbrace{(-2-\lambda)\cdot\left((2-\lambda)(1-\lambda)-0\right)} = 0$$

$$2\cdot\underline{(2-\lambda)} + (-2-\lambda)\underline{(2-\lambda)}(1-\lambda) = 0$$

$$(2-\lambda)\cdot\left[2 + (-2-\lambda)(1-\lambda)\right] = 0$$

$$(2-\lambda)\cdot(\lambda^2+\lambda) = 0$$

$\underline{\lambda = 2}$ or $\lambda^2 + \lambda = 0$

$\lambda = \underline{0}$, $\lambda = \underline{-1}$

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4.10. $A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}$ $\underline{v} =$

a) $|A| = s \cdot (4+2) = \underline{6s}$

$\text{rk}\,A = \begin{cases} 3, & s \neq 0 \\ 2, & s = 0 \end{cases}$

since $|A| = 0$, we have $\text{rk}\,A < 3$

$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 4 \end{pmatrix}$

$M_{13,12} = \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} = -6 \neq 0$

b) Eigenvalues of A:

$\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$

$(s-\lambda) \cdot \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = 0$

$(s-\lambda) \cdot (\lambda^2 - 5\lambda + 6) = 0$

$\underline{\lambda = s}$ or $\lambda^2 - 5\lambda + 6 = 0$

$\cancel{\lambda = 2, 3}$

$\underline{\lambda = 2}, \ \underline{\lambda = 3}$

c) Diagonalizable:

$D = \begin{pmatrix} s & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P = (\underline{v_1} | \underline{v_2} | \underline{v_3})$

$s \neq 2, 3$:

All eigenvalues are distinct $(m=1)$ ok

$s = 2$:

$\boxed{\lambda_1 = \lambda_2 = 2 \quad m = 2}$

$\lambda_3 = 3$ ok $m = 1$

$s = 2, \lambda = 2$:

$\begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \sim \left(\begin{array}{ccc|c} -1 & 7 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{array}\right)$

z free

$\boxed{\text{1 free var} \quad m = 2}$

A not diagon. for $\underline{s = 2}$

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$\lambda = 2$ ok.

$s=3$: $\lambda = 3, 2, 3$
$= 3$, (mult 2)
2 (mult 1)

Check: $s=3, \lambda=3$ $m=2$ $\Downarrow$ need 2 free variables.

$$\begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

one free var (z)
pivots for x, y
$\Downarrow$
A not diag. for $s=3$

Markov chains

$A = (a_{ij})$ $\boxed{\begin{array}{l} a_{ij} \geq 0 \\ \sum \text{col.} = 1 \end{array}}$

$a_{ij} > 0 \Rightarrow A$ regular

Long run equilibrium: The unique eigenvector for $\lambda = 1$ with entries that has sum $= 1$.

Midterm 03/2016, Pb 6:

$$A = \begin{pmatrix} 0.65 & 0.21 \\ 0.35 & 0.79 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0.65-1 & 0.21 \\ 0.35 & 0.79-1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-0.35x + 0.21y = 0 \iff \begin{pmatrix} -0.35 & 0.21 \\ \cancel{0.35} & \cancel{-0.21} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-35x + 21y = 0$
$\uparrow$ 21, $\uparrow$ 35

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \cdot \begin{pmatrix} 21 \\ 35 \end{pmatrix} = \begin{pmatrix} 21t \\ 35t \end{pmatrix} = \begin{pmatrix} 21/56 \\ 35/56 \end{pmatrix}$$

$21t + 35t = 56t = 1$

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5.9

$$A = \begin{pmatrix} 0.61 & 0.13 & 0.03 \\ 0.13 & 0.81 & 0.10 \\ 0.26 & 0.06 & 0.87 \end{pmatrix}$$

$$\begin{pmatrix} -0.39 & 0.13 & 0.03 \\ 0.13 & -0.19 & 0.10 \\ 0.26 & 0.06 & -0.13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

eigenv. for $\lambda = 1$
$\Downarrow$
one free var.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \cdot \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

Equilibrium: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s.t. $x+y+z=1$

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c) Definiteness:

$D_1, D_2, \ldots, D_n > 0$: positive def.
$D_1 < 0, D_2 > 0, D_3 < 0, \ldots$: negative definite

(even) $D_2, D_4, \ldots < 0$
or
(odd) D_1, D_3, D_5 of different signs
} : indefinite

otherwise: compute all Δ_i principal minor

Local max/min:

FOC: $f'_x = f'_y = \ldots = 0 \rightarrow$ Solutions = Stationary pts.

SOC: $\underline{x}^*$ stationary pt:

$H(f)(\underline{x}^*)$ pos. def. : $\underline{x}^*$ local ~~max~~ min
" neg. def. : $\underline{x}^*$ local ~~min~~ max
" indefinite : $\underline{x}^*$ saddle pt

Convex/concave:

f convex $\Leftrightarrow$ $H(f)(\underline{x})$ pos. semidefinite for all pts. $\underline{x}$

f concave $\Leftrightarrow$ $H(f)(\underline{x})$ neg. semidef. for all pts. $\underline{x}$.

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Midterm 10/2015:

(7) $f = 2x_1^2 + 6x_1x_2 + 5x_2^2 - 2x_2x_3 + 3x_3^2 + 2x_3x_4 + 4x_4^2$

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 5 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \qquad \left.\begin{matrix} D_1 = 2 \\ D_2 = 1 \\ D_3 = 1 \\ D_4 = 3 \end{matrix}\right\} \text{pos. defn.} \quad \text{(B)}$$

$$D_3 = +1 \cdot (-2) + 3 \cdot 1 = 1$$

$$D_4 = -1 \cdot \begin{vmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & -1 & 1 \end{vmatrix} + 4 \cdot D_3$$

$$= -(1 \cdot 1) + 4 \cdot 1 = 3$$

(8) $a > 0$, $f = x^a\sqrt{y} = x^a y^{1/2}$, $x, y > 0$

$$f'_x = a \cdot x^{a-1} y^{1/2}$$

$$f'_y = \frac{1}{2} \cdot x^a y^{-1/2}$$

$$H(f) = \begin{pmatrix} a(a-1)x^{a-2}y^{1/2} & \frac{1}{2}ax^{a-1}y^{-1/2} \\ \frac{1}{2}ax^{a-1}y^{-1/2} & -\frac{1}{4}x^a y^{-3/2} \end{pmatrix}$$

$$D_1 = a \cdot (a-1)x^{a-2}y^{1/2} \qquad \text{pos } a > 1 \qquad \text{neg } a < 1$$

$$D_2 = -\frac{1}{4}a(a-1)x^{2a-2}y^{-1} - \frac{1}{4}a^2x^{2a-2}y^{-1}$$

$$= -\frac{1}{4}ax^{2a-2}y^{-1}\underbrace{(a-1+a)}_{2a-1} \qquad \begin{matrix} \text{pos} & a < 1/2 \\ \text{neg} & a > 1/2 \end{matrix}$$

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$a > 1/2$: $D_2 < 0$ indef. $\Rightarrow$ f not convex, not concave

$a < 1/2$: $D_2 > 0$, $D_1 < 0$ neg. defn. $\Rightarrow$ f concave

(C)

(4) $A = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

$\begin{vmatrix} 1-\lambda & \sqrt{2} & 0 \\ \sqrt{3} & 1-\lambda & 0 \\ 0 & 0 & -6-\lambda \end{vmatrix} = (-6-\lambda)\cdot(\lambda^2 - 2\lambda + (1-\sqrt{6})) = 0$

$\lambda = -6$, $\lambda = \frac{2 \pm \sqrt{\cancel{4} - 4(\cancel{1} - \sqrt{6})}}{2}$

one neg.

$= \frac{2 \pm \sqrt{4\sqrt{6}}}{2}$

(C) One pos. One neg.

(5) $A = \begin{pmatrix} 1 & s & s \\ 0 & 2 & s \\ 0 & 0 & 3 \end{pmatrix}$

$\lambda = 1, 2, 3$
All distinct $\Rightarrow$ diag. for all s, (A)

(6) $A = \begin{pmatrix} 0.74 & 0.13 \\ 0.26 & 0.87 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -0.26 & 0.13 \\ 0.26 & -0.13 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = t\cdot\begin{pmatrix} 13 \\ 26 \end{pmatrix} = \begin{pmatrix} 13/39 \\ 26/39 \end{pmatrix}$

$x + y = 1$: $13t + 26t = 39t = 1$
$t = 1/39$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$

(D)

(1)

$$\left(\begin{array}{ccccc|c} \ast & \ast & \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & \ast & \ast & 0 \\ \ast & \ast & \ast & \ast & \ast & 0 \end{array}\right)$$

(D)

3 free $\Rightarrow$ inf. many solutions, three free var's.

$n - rk(A) = 5 - 2 = \underline{\underline{3}}$

(2)

$\underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ s \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \\ s & 1 & 3 \end{pmatrix}$

$|A| = 1 \cdot 8 - 2 \cdot (-3 - s) + 4 \cdot (-1 - 3s)$

$= 8 + 6 + 2s - 4 - 12s$

$= \underline{10 - 10s}$

$rk\, A = \begin{cases} 3, & s \neq 1 \\ <3, & s = 1 \end{cases}$

lin. independent $\Leftrightarrow rk\, A = 3 \Leftrightarrow |A| \neq 0 \Leftrightarrow \underline{\underline{s \neq 1}}$

(3)

$A = \begin{pmatrix} 1 & 4 & -7 & 3 \\ 3 & 2 & 1 & 3 \\ 4 & 6 & t & 1-t \end{pmatrix}$ -3 -4

$\rightarrow \begin{pmatrix} 1 & 4 & -7 & 3 \\ 0 & -10 & 22 & -6 \\ 0 & -10 & 28+t & -11-t \end{pmatrix}$ -1

$\rightarrow \begin{pmatrix} 1 & 4 & -7 & 3 \\ 0 & -10 & 22 & -6 \\ 0 & 0 & 6+t & -5+t \end{pmatrix}$

$t \neq -6: rk\, A = 3$

$t = -6: rk\, A = 3$

(D)

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Midterm 04/2015, Pb 7.

$$f = x^4 + 4xy + y^4$$

$$f'_x = 4x^3 + 4y = 0 \qquad x^3 + y = 0 \qquad y = -x^3$$
$$f'_y = 4x + 4y^3 = 0 \qquad x + y^3 = 0$$

$$x + (-x^3)^3 = 0$$
$$x - x^9 = 0$$
$$\underline{x} \cdot (1 - x^8) = 0$$
$$x = 0, \quad x^8 = 1$$
$$x = \pm 1$$

$$\left.\begin{array}{l}(x,y) = (0,0)\\ \qquad\quad (1,-1)\\ \qquad\quad (-1,1)\end{array}\right\} \text{Stat. pts.}$$

$$H(f) = \begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}$$

$$H(f)(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \quad D_1 = 0, \ \underline{D_2 = -16} \quad \text{indefinite} \quad \underline{\text{saddle}}$$

$$H(f)(1,-1) = \begin{pmatrix} 12 & 4 \\ 4 & 12 \end{pmatrix} \quad D_1 = 12, \ D_2 = 144 - 16 = 128 \quad \text{pos. defn.} \quad \underline{\text{local min}}$$

$$H(f)(-1,1) \quad // \quad \underline{\text{local min}}$$

B